Cohomological Invariants
for Higher Degree Forms

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Christopher Rupprecht
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0 Introduction

The motivation for this work is to generalize a concept from the theory of quadratic forms to higher degree forms. Let us first recall some definitions for quadratic forms (For a detailed exposition, see e.g. [28]):

Let K be a field of characteristic $\neq 2$. A *quadratic form over* K is a pair (V, b), consisting of a finite-dimensional K-vector space V and a symmetric bilinear form $b : V \times V \to K$. The set of isomorphism classes of non-degenerate quadratic forms over K with direct sum and tensor product is a semiring, which embeds into a commutative K-algebra $\hat{W}(K)$, the *Witt-Grothendieck ring of quadratic forms over* K. The 2-dimensional quadratic form $h = \langle 1, -1 \rangle \in \hat{W}(K)$ is called the *hyperbolic plane*, and the ideal $H \subset \hat{W}(K)$ generated by h is the *ideal of hyperbolic forms*. The quotient ring $W(K) = \hat{W}(K)/H$ is the *Witt ring of quadratic forms over* K. The structure of this ring is the principal object of study in the theory of quadratic forms.

The dimension map $\dim : \hat{W}(K) \to \mathbb{Z}$, $(V, b) \mapsto \dim_K(V)$ induces a homomorphism $e_0 : W(K) \to \mathbb{Z}/2$, called the *dimension index*. Let $I = I(K) \subset W(K)$ be its kernel, called the *fundamental ideal* of the Witt ring. The filtration of the Witt ring by the powers of the fundamental ideal relates the Witt ring of quadratic forms to Milnor K-Theory and Galois cohomology of the field K as follows:

Let $K_n^{\mathrm{M}}(K)$ be the n-th Milnor K-group of the field K, defined by Milnor in [25]. In this article, Milnor also gives a a surjection $s_n : K_n^{\mathrm{M}}(K) \to I^n/I^{n+1}$, which maps a product $l(a_1) \cdots l(a_n)$ to the class of the n-fold Pfister form $(\langle a_1 \rangle - \langle 1 \rangle) \cdots (\langle a_n \rangle - \langle 1 \rangle)$. Milnor's conjecture that s_n is an isomorphism was proved by Orlov, Vishik and Voevodsky in [26].

For $r \geq 2$, we have $K_1^{\mathrm{M}}(K)/r \cong K^*/K^{*r}$, and in [31], Tate shows that the Kummer isomorphism $K^*/K^{*r} \xrightarrow{\sim} H^1(K, \mu_r)$ extends to a homomorphism $h_{n,r} : K_n^{\mathrm{M}}(K) \to H^n(K, \mu_r^{\otimes n})$ via the cup product. In ([19], p.608), Kato conjectures that $h_{n,r}$ is bijective. In the case $r = 2$, this had been conjectured earlier by Milnor and by Bloch. The conjecture was proved by Voevodsky in the case that $r = 2^m$ is a power of 2 (cf. [17]). Hence we obtain commutative diagrams of abelian groups and isomorphisms

$$\begin{tikzcd}
K_n^{\mathrm{M}}(K)/2 \arrow[rr, "h_n"] \arrow[dr, "s_n"'] & & H^n(K, \mu_2^{\otimes n}) \\
& I^n/I^{n+1}. \arrow[ur, "e_n"'] &
\end{tikzcd}$$

For $n = 0, 1, 2$, the morphism e_n has the following interpretation in terms of quadratic forms: For $n = 0$, this is the dimension index e_0, which was defined above.

The morphism e_1 is defined as follows: For a quadratic form(V, b), the class of the determinant $\det(V, b)$ in K^*/K^{*2} is an invariant for its isomorphism class.

The discriminant of (V,b) is defined as $d(V,b) := (-1)^{\lfloor \frac{dim(b)}{2} \rfloor}\det(V,b)$ (cf. [28], Def. 2.2.1). The discriminant gives a morphism $d : I \to K^*/K^{*2} \cong H^1(K,\mu_2)$, and e_1 is the induced map on I/I^2.

The morphism e_2 is given by the Clifford invariant, which maps a quadratic form to the class of its Clifford algebra in the Brauer group. This class has degree 2, so that the image of e_2 lies in $\mathrm{Br}(K)_2 \cong H^2(K,\mu_2) \cong H^2(K,\mu_2^{\otimes 2})$ (cf. [22], Chap. 5.3).

Independently from the proof of the Milnor conjecture, it was shown for $n = 3$ by Arason in [1] and for $n = 4$ by Jacob and Rost in [15] that the map e_n completing the diagram is well defined.

Now let $r > 2$ be an integer. One observes that, while the upper part of the diagram has a degree r analogue, the lower part has not:

$$\begin{tikzcd}
K_n^{\mathrm{M}}(K)/r \arrow[rr, "h_{n,r}"] \arrow[dr, "s_{n,r}?"'] & & H^1(K,\mu_r^{\otimes n}) \\
& I^n/I^{n+1}? \arrow[ur, "e_{n,r}?"'] &
\end{tikzcd}$$

This raises the following questions:

- Is there a degree r analogue of the Witt-Grothendieck ring?
- Can we give cohomological invariants for higher degree forms generalizing the maps e_n in the diagram above?
- Can we give a degree r analogue of the hyperbolic plane or the hyperbolic ideal and define a Witt ring of higher degree forms?
- Can we give degree r Pfister forms generalizing the maps s_n in the diagram above?

Forms of degree r. Let K be a field such that $(\mathrm{char}(K), r!) = 1$, i.e. such that $\mathrm{char}(K) = 0$ or $\mathrm{char}(K) > r$. An r-form over K is a pair (V,Θ), consisting of a finite-dimensional K-vector space V and a symmetric multilinear map $\Theta : V \times \cdots \times V \to K$, defined on the r-fold product of V.

The condition $(\mathrm{char}(K), r!) = 1$ on the characteristic of K allows us to identify r-forms with homogeneous forms of degree r over K as follows: Let (V,Θ) be an r-form over K, and let $\{v_1,\ldots,v_n\}$ be a K-basis of V. Then there is a homogeneous form $f = f_\Theta \in K[x_1,\ldots,x_n]$ such that

$$f(x_1,\ldots,x_n) = \Theta(\sum_{i=1}^n x_i v_i, \ldots, \sum_{i=1}^n x_i v_i).$$

Just as in the case of quadratic and bilinear forms, we obtain a bijective correspondence between isomorphism classes of symmetric multilinear r-forms and homogeneous forms of degree r. In times it will be convenient to switch from one viewpoint to the other. We will speak of multilinear and homogeneous r-forms, or simply of r-forms if there is no ambiguity.

Regularity. A quadratic form on V is called non-degenerate if the induced linear map $V \to V^*$ has full rank. A quadratic form is non-degenerate if and only if it is non-singular, meaning that it describes a non-singular quadric. For forms of degree $r > 2$, there is more than one analogue of this definition:

Definition. Let $r \geq 2$ and let $1 \leq k < r$ be an integer. An r-form (V, Θ) over K is called *k-regular*, if, for every non-zero k-tuple $(v_1, \ldots, v_k)$ of vectors in V, the $(r-k)$-form $(V, \Theta_{(v_1,\ldots,v_k)})$ given by $\Theta_{(v_1,\ldots,v_k)}(v_{k+1}, \ldots, v_r) := \Theta(v_1, \ldots, v_r)$ is non-zero. A 1-regular r-form is also called *regular*.

An r-form over K is non-singular, meaning that it describes a non-singular hypersurface in projective space, if and only if it is $(r-1)$-regular over the separable closure $\bar{K}$.

The Witt-Grothendieck ring of r-forms. The starting point for this work is the article [10], in which Harrison introduces a ring of r-forms. He shows that the set of isomorphism classes of regular r-forms over K with direct sum and tensor product is a commutative semiring over K, which embeds into a commutative K-algebra $\hat{W}_r(K)$, called the Witt-Grothendieck ring of r-forms.

Although the definition of the Witt-Grothendieck ring of r-forms is the same for $r = 2$ and $r > 2$, the obtained rings have quite different properties. This is illustrated by the following observations:

Consider the generators in the Witt-Grothendieck ring. Every quadratic form is isomorphic to a diagonal form, and therefore the Witt-Grothendieck ring of quadratic forms is generated by 1-dimensional forms. In particular, the Witt-Grothendieck ring of quadratic forms over a finite field is finitely generated.

Forms of degree $r > 2$ are not always diagonal. We call an r-form *indecomposable* if it has no non-trivial sum decomposition. Over any field, there are indecomposable r-forms of dimension > 1. If K is a finite field, then there are indecomposable r-forms of arbitrary dimension over K, and the Witt-Grothendieck ring of r-forms over K is not finitely generated.

Now consider the relations in the Witt-Grothendieck ring. Witt's Theorem gives a cancellation rule for quadratic forms, which allows the construction of the Witt-Grothendieck group. For $r > 2$ one obtains a stronger result: The decomposition of an r-form into indecomposable r-forms is unique. Thus, the Witt-Grothendieck group of degree $r > 2$ is a free abelian group, having much less relations than in the quadratic case.

Separable r-forms. Another difference between the quadratic and the degree $r > 2$ case comes from the following definition given by Harrison:

Definition. Let $r > 2$, and let (V, Θ) be an r-form over K. Let the center of (V, Θ), written $\mathrm{Cent}_K(V, \Theta)$, denote the set of K-endomorphisms $\varphi \in \mathrm{End}_K(V)$ such that

$$\Theta(\varphi v_1, v_2, v_3, \ldots, v_r) = \Theta(v_1, \varphi v_2, v_3, \ldots, v_r)$$

for all $v_1, \ldots, v_r \in V$. The center is a commutative K-algebra. The r-form (V, Θ) over K is called *separable* if its center is a separable K-algebra such that $\dim_K(\mathrm{Cent}(V, \Theta)) = \dim_K(V)$.

Harrison shows that separable r-forms generate a subring $\hat{W}_r^{sep}(K) \subset \hat{W}_r(K)$ in the Witt-Grothendieck ring of r-forms, and he gives the following classification of separable r-forms:

Let L/K be a finite separable field extension, let $\mathrm{tr}_{\mathrm{L/K}} : L \to K$ be the trace map, and let $b \in L^*$. We consider L as a K-vector space with the multilinear map

$$\mathrm{tr}_{\mathrm{L/K}}\langle b \rangle_r : L \times \cdots \times L \to K \quad , \quad (l_1, \ldots, l_r) \mapsto \mathrm{tr}_{\mathrm{L/K}}(b l_1 \cdots l_r).$$

Then $(L, \mathrm{tr}_{\mathrm{L/K}}\langle b \rangle_r)$ is an indecomposable separable r-form over K and every indecomposable separable r-form over K is isomorphic to an r-form $(L, \mathrm{tr}_{\mathrm{L/K}}\langle b \rangle_r)$ for some L and b.

Cohomological classification of separable r-forms. In the theory of quadratic forms, Weil descent is used to classify quadratic forms by Galois cohomology: Since every quadratic form is diagonal, all quadratic forms of the same dimension over K are isomorphic over a separable closure $\bar{K}$. Therefore the set of quadratic forms of dimension n over K is bijective to the cohomology set $H^1(K, \mathrm{O}_n)$ by Weil descent.

For $r > 2$, it is not true that all r-forms become isomorphic to a diagonal form over the separable closure. However, restricting attention to those who do so, we obtain a subring in the Witt-Grothendieck ring, and we find that this is the ring of separable r-forms. This leads to a cohomological classification for separable r-forms as follows:
The automorphism group of the diagonal r-form over $\bar{K}$ is the wreath product $S_n \int \mu_r$ of the symmetric group S_n and the group μ_r of r-th roots of unity in $\bar{K}$. The wreath product is the set $S_n \times \mu_r^{\oplus n}$ with the semidirect product induced by S_n-action on $\mu_r^{\oplus n}$. Using Weil descent, we obtain a classification of separable r-forms of dimension n over K by the cohomology set $H^1(K, S_n \int \mu_r)$. The correspondence between this classification and Harrison's classification by trace forms is explicitly computed.

Cohomological invariants for separable r-forms. Consider the classification of quadratic forms by cohomology sets $H^1(K, \mathrm{O}_n)$, which was described before. In these terms, the determinant of quadratic forms, which is closely related to the map e_1 in the diagram above, is equal to the cohomology map $H^1(K, \mathrm{O}_n) \to H^1(K, \mu_2)$. In the same way, we obtain invariants for separable r-forms from the cohomological classification:

Consider the projection from the wreath product $S_n \int \mu_r$ to the symmetric group S_n. The cohomology set $H^1(K, S_n)$ classifies isomorphism classes of separable K-algebras of dimension n, and we find that the induced cohomology map $H^1(K, S_n \int \mu_r) \to H^1(K, S_n)$ maps a separable r-form to the isomorphism class of its center. Concatenation with the sign homomorphism $S_n \to \mu_2$ gives an invariant map $H^1(K, S_n \int \mu_r) \to H^1(K, \mu_2)$, which maps a separable r-form to the determinant of the bilinear trace form of its center in $K^*/K^{*2} \cong H^1(K, \mu_2)$.

Next, consider the permanent morphism

$$\mathrm{per} : S_n \int \mu_r \to \mu_r \ , \quad (\sigma, (\alpha_1, \ldots, \alpha_n)) \mapsto \prod_{i=1}^{n} \alpha_i \ .$$

The induced cohomology map gives a first degree cohomological invariant for separable r-forms

$$\mathrm{per} : \hat{W}_r^{sep}(K) \to H^1(K, \mu_r).$$

Finally, consider the determinant morphism

$$\det : S_n \int \mu_r \to \bar{K}^* \ , \quad (\sigma, (\alpha_1, \ldots, \alpha_n)) \mapsto \mathrm{sgn}(\sigma) \cdot \prod_{i=1}^{n} \alpha_i \ .$$

The image of the determinant is equal to μ_r if r is even, and equal to μ_{2r} if r is odd. Hence we obtain another first degree cohomological invariant for separable r-forms

$$\det : \hat{W}_r^{sep}(K) \to \left\{ \begin{array}{c} H^1(K, \mu_r) \\ H^1(K, \mu_{2r}) \end{array} \right\} \text{ if } r \text{ is } \left\{ \begin{array}{c} \text{even} \\ \text{odd} \end{array} \right\}.$$

All these invariants are given explicitly in terms of separable trace forms.

Cohomological invariants of degree 2. We interpreted the determinant of quadratic forms as a cohomology map of degree 1, induced by the determinant morphism $\det : \mathrm{O}_n \to \mu_2$. Its kernel is the special orthogonal group SO_n, hence the cohomology set $H^1(K, \mathrm{SO}_n)$ classifies quadratic forms of dimension n and determinant 1. The simply-connected covering of SO_n is the spin group

$$0 \to \mu_2 \to \mathrm{Spin}_n \to \mathrm{SO}_n \to 0,$$

and the induced long exact sequence of Galois cohomology gives a map

$$\delta : H^1(K, \mathrm{SO}_n) \to H^2(K, \mu_2),$$

which is related to the morphism e_2 in the diagram above (cf. [21], §2.4).

Starting from the cohomological classification for separable r-forms, we want to construct second degree invariants in this way. For this purpose, let $\mathrm{SO}_{r,n}^{(i)} \in S_n \int \mu_r$, $(i = 1, 2, 3)$ denote the kernel of the permanent, the determinant, and the sign respectively. In the case that $r \neq 2, 3$ is a prime number, we give a classification for central extensions of Galois modules

$$0 \to \mu_r \to \mathrm{Spin}_{n,r} \to \mathrm{SO}_{n,r}^{(i)} \to 0.$$

We find that there is a canonical extension $\mathrm{Spin}_{n,r}$ of the group $\mathrm{SO}^{(3)}_{n,r}$, while the other two groups have only the trivial extension. This leads to a second degree cohomological invariant

$$\delta : H^1(K, \mathrm{SO}^{(3)}_{n,r}) \to H^2(K, \mu_r)$$

for r-forms of dimension n and sign 1. However, δ vanishes for r-forms of trivial permanent, so this does not lead to a new classification result.

In the following parts, we examine several other invariants for r-forms and their relations with the previous ones.

The generalized Leibniz formula. Let Θ be an r-form on the K-vector space V and let $\{v_1, \ldots, v_n\}$ be a K-basis of V. We consider a generalization of the Leibniz formula for the quadratic determinant:

$$\det{}'(\Theta) := \sum_{\sigma_2, \ldots, \sigma_r \in S_n} \mathrm{sgn}(\sigma_2 \cdots \sigma_r) \prod_{i=1}^{n} \Theta(v_i, v_{\sigma_2 i}, \ldots, v_{\sigma_r i}).$$

This formula has first been studied in the 19th century. In ([5], p.86), Cayley found that, if r is even, $\det'$ induces an invariant map

$$\det{}' : \hat{W}_r(K) \to K/K^{*r}.$$

For r-forms of odd degree, however, this formula does not give a well-defined invariant. We find the following

Theorem(6.4). Let r be even. Then $\det' = \det : \hat{W}^{sep}_r(K) \to K^*/K^{*r}$.

The discriminant. Another invariant for r-forms is the discriminant. Given a homogeneous r-form

$$f(x_1, \ldots, x_n) = \sum_{\nu_1 + \cdots + \nu_n = r} a_{\nu_1 \ldots \nu_n} x_1^{\nu_1} \cdots x_n^{\nu_n}$$

with coefficients $a_\nu \in K$, the discriminant $\Delta(f)$ is a polynomial expression in the a_ν, which vanishes if and only if f is non-singular. The discriminant induces an invariant map $\Delta_r : \hat{W}_r(K) \to K/K^{*r}$. We obtain the following

Theorem(7.4). Let Θ be a separable r-form of dimension n over K. Then

$$\Delta_r(\Theta) = \left\{ \begin{array}{c} \det(\Theta)^{(-1)^{n-1}} \\ \mathrm{per}(\Theta)^{(-1)^{n-1}} \end{array} \right\} \in K^*/K^{*r} \text{ if } r \text{ is } \left\{ \begin{array}{c} \text{even} \\ \text{odd} \end{array} \right\}.$$

For non-separable r-forms, however, there seems to be no relation between the discriminant and the other invariants. This is shown at the example of hyperelliptic curves.

The hyperdeterminant. Defined similarly as the discriminant, the hyperdeterminant is an invariant for arbitrary multilinear forms of degree r, where 'arbitrary' means including non-symmetric forms. Let $\Theta : V \times \cdots \times V \to K$ be such a form, and let $\{v_1, \ldots, v_n\}$ be a basis of V. Let f be the homogeneous r-form in $n \cdot r$ variables given by

$$f(x_1^{(1)}, \ldots, x_n^{(1)}, \ldots, x_1^{(r)}, \ldots, x_n^{(r)}) := \Theta(\sum_{i=1}^{n} x_i^{(1)} v_i, \ldots, \sum_{i=1}^{n} x_i^{(r)} v_i).$$

The hyperdeterminant is a polynomial expression in the coefficients of f which vanishes if and only if f is non-singular.

In the case of quadratic forms, the hyperdeterminant is equal to the discriminant. In degree $r > 2$, however, a simple computation shows that the hyperdeterminant vanishes for diagonal r-forms of dimension $n \geq 4$. Hence there are no classification results to be expected from this invariant in our sense.

Zeta functions of separable r-forms over a finite field. In the theory of quadratic forms, recently the motives corresponding to the induced varieties have become an object of study, and therefore it seems appropriate to ask whether the determinant of an r-form just depends on the corresponding motive. Let K be a finite field, let Θ be an r-form of dimension n over K, and let $X \subset \mathbb{P}_K^{n-1}$ be the projective hypersurface described by Θ. The zeta function of X is defined as

$$\zeta(\Theta, t) = \zeta(X, t) = exp(\sum_{i \geq 1} \tfrac{\nu_i}{i} t^i) \in \mathbb{Q}(t),$$

where $\nu_i := \mathrm{card}(X(\mathbb{F}_{q^i}))$ is the number of F_{q^i}-rational points of X. The zeta function of X is an invariant of the motive corresponding to X, and the Tate conjecture implies that it determines the motive. Thus, if we assume that the determinant of r-forms gives an invariant for the induced motives over k, then we would expect that r-forms with equal zeta function should have equal determinant. However, the following argument shows that we can not not expect too much: The zeta function is a projective invariant, hence it remains unchanged if we exchange Θ by a multiple $a\Theta$ with $a \in K^*$. But this changes the determinant by the factor a^n, hence its class in K^*/K^{*r} is changed if the dimension n is not a multiple of the degree r. In fact, we find the following

Theorem(8.11). Let K be a finite field such that the prime field contains the r-th roots of unity. Let n be a multiple of r and let Θ and Ψ be separable r-forms of dimension n over K having the same zeta function. Then Θ and Ψ have the same determinant.

The proof relies on an explicit formula given by Brünjes in [2] for the zeta function of such r-forms. Deligne shows that its coefficients are algebraic integers in the r-th cyclotomic field $\mathbb{Q}(\zeta_r)$, and André Weil computes their prime decomposition, which allows an analysis of Brünjes's formula proving the Theorem.

Hyperbolic forms of degree r and the Witt ring. In order to give a definition for a Witt ring of r-forms, we want to find a degree r analogue of the hyperbolic ideal in the Witt-Grothendieck ring. This Lemma describes what we are out for:

Lemma(9.1). Let $d : \hat{W}_r^{sep}(K) \to K^*/K^{*r}$ be the permanent or the determinant. Let $H \subset \hat{W}_r^{sep}(K)$ be an ideal such that $\dim(H) \equiv 0$ modulo r and $d(H) = 1$. Let $W_r(K) := \hat{W}_r^{sep}(K)/H$ and let $I_r \subset W_r(K)$ denote the kernel of the dimension index $\dim : W_r(K) \to \mathbb{Z}/r$. Then d induces a surjective morphism

$$d : I_r/I_r^2 \to K^*/K^{*r}.$$

Having in mind the diagram we went out from, we would expect that this map is an isomorphism for the right choice of the invariant d and the ideal H. In the case that $r \neq 2$ is a prime number, we propose a degree r analogue of the hyperbolic plane $h_2 = \langle 1, -1 \rangle$:

Definition(9.2). Let $\phi := x^{r-1} + \cdots + x + 1 \in K[x]$ be the r-th cyclotomic polynomial and let L denote the separable K-algebra $K[x]/(\phi)$. Let h_r be the r-form

$$h_r := \langle 1 \rangle_r \oplus (L, \mathrm{tr}_{\mathrm{L/K}} \langle x \rangle_r),$$

where $\langle 1 \rangle_r$ denotes the 1-dimensional r-form x^r and $(L, \mathrm{tr}_{\mathrm{L/K}} \langle x \rangle_r)$ is the trace form of degree r on the K-vector space L given by $(l_1, \ldots, l_r) \mapsto \mathrm{tr}_{\mathrm{L/K}}(x l_1 \cdots l_r)$.

With this definition, h_r is a separable r-form of dimension r and permanent 1. If the field K contains a primitive r-th roots of unity ζ, then h_r is isomorphic to the diagonal r-form $x_1^r + \zeta x_2^r + \cdots + \zeta^{r-1} x_r^r$.

In order to test this definition, we let H be the ideal generated by h_r and compute the group I/I^2 in the case that K is a finite field. We find, however, that this group is not even finitely generated, hence the permanent is far from giving an isomorphism here. This indicates that we have to choose the ideal of hyperbolic r-forms much bigger.

Pfister forms of degree r. Looking at the last of the four questions posed in the beginning, we face a major problem: The Milnor K-algebra is a graded anticommutative ring, in the sense that we have $xy = (-1)^{mn}$ for $x \in K_m^{\mathrm{M}}$ and $y \in K_n^{\mathrm{M}}$. In the case $r = 2$, this was no problem, since we only considered K-groups modulo 2. For $r > 2$, however, we can not expect to find an analogue of the definition of the Milnor isomorphism generated by Pfister forms of degree 1, as long as our Witt-Grothendieck ring is commutative. This is clearly the case, so that we have to leave this questions open.

Contents of the sections. The sections of this thesis are organized as follows: The first section gives the basic concepts for the work with r-forms, including the definition of the Witt-Grothendieck ring.

Section 2 introduces the language of homogeneous r-forms and gives translations between multilinear and homogeneous r-forms.

Section 3 introduces the center of r-forms and the ring of separable r-forms with its classification by trace forms.

Section 4 studies the cohomological classification of separable r-forms using descent and gives first degree cohomological invariants for these.

Section 5 asks for the existence of second degree cohomological invariants.

Section 6 examines the generalized Leibniz formula.

Section 7 introduces the discriminant and the hyperdeterminant of r-forms and examines their relations with the invariants defined in section 4.

Section 8 studies zeta functions of separable r-forms over a finite field and their relation with the determinant.

Section 9 proposes a definition for the hyperbolic plane of degree r and computes the induced Witt ring.

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1 The Witt-Grothendieck Ring of r-Forms

This section gives the basic notations and definitions for our work with r-forms. Most of the material presented here is due to the work of Harrison and Pareigis in [10] and [11], and it is included here without further reference. All statements are formulated for r-forms over a field, though it is not difficult to generalize many of them to r-forms over rings (cf. [10],[11]). Here we use the language of multilinear r-forms, and the next section will deal with the equivalent statements for homogeneous r-forms.

Throughout this thesis, let $r \geq 2$ be an integer and let K be a field in which $r!$ is invertible.

1.1 Definition. (r-Forms over K) *An r-form over K is a pair (V, Θ), consisting of a finite-dimensional K-vector space V and a symmetric K-multilinear map $\Theta : V^r \to K$. An isomorphism $t : (V, \Theta) \to (W, \Psi)$ of r-forms is an isomorphism of K-vector spaces $t : V \to W$ such that $\Psi(tv_1, \ldots, tv_r) = \Theta(v_1, \ldots, v_r)$ for all $v_1, \ldots, v_r \in V$.*

1.2 Definition. *Let (V, Θ) be an r-form over K.*

(i) For $1 \leq k < r$ and $v_1, \ldots, v_k \in V$, let $\Theta_{(v_1,\ldots,v_k)}$ denote the $(r-k)$-form given by $\Theta_{(v_1,\ldots,v_k)}(v_{k+1}, \ldots, v_r) := \Theta(v_1, \ldots, v_r)$.

(ii) An r-form (V, Θ) is called k-regular if, for every non-zero k-tuple $(v_1, \ldots, v_k)$ of vectors in V, the $(r-k)$-form $(V, \Theta_{(v_1,\ldots,v_k)})$ is non-zero. A 1-regular r-form is called regular.

Remark. It is not difficult to see that an r-form Θ is k-regular if and only if, for every non-zero $v \in V$, the $(r-k)$-form $(V, \Theta_{(v,\ldots,v)})$ is non-zero. The proof can be found in Lemma 2.1 in the next section.

1.3 Definition. (Sums and Products of r-Forms, Scalar Extension)

(i) Let (V, Θ) and (W, Ψ) be r-forms over K. For $v_i \in V$ and $w_i \in W$ let

$$(\Theta \oplus \Psi)(v_1 \oplus w_1, \ldots, v_r \oplus w_r) := \Theta(v_1, \ldots, v_r) + \Psi(w_1, \ldots, w_r)\,.$$

Let $(V, \Theta) \oplus (W, \Psi)$ denote the r-form $(V \oplus W, \Theta \oplus \Psi)$ over K.

(ii) Let (V, Θ) and (W, Ψ) be r-forms over K. For $v_i \in V$ and $w_i \in W$ let

$$(\Theta \otimes \Psi)(v_1 \otimes w_1, \ldots, v_r \otimes w_r) := \Theta(v_1, \ldots, v_r) \cdot \Psi(w_1, \ldots, w_r)\,.$$

Let $(V, \Theta) \otimes_K (W, \Psi)$ denote the r-form $(V \otimes_K W, \Theta \otimes \Psi)$ over K.

(iii) Let L/K be a field extension. For $v_i \in V$ and $l_i \in L$, let

$$\Theta_L(v_1 \otimes l_1, \ldots, v_r \otimes l_r) := l_1 \cdots l_r\, \Theta(v_1, \ldots, v_r)\,.$$

Let $(V, \Theta)_L$ denote the r-form $(V \otimes_K L, \Theta_L)$ over L.

(iv) Let L/K be a finite field extension, let (U, Φ) be an r-form over L, and let $t \in \mathrm{Hom}_K(L, K)$ be non-trivial. Consider U as a K-vector space with the K-multilinear map

$$t \circ \Phi : U \times \cdots \times U \to K.$$

Let $t(U, \Phi)$ denote the r-form $(U, t \circ \Phi)$ over K.

1.4 Notations and Examples.

(i) An r-form is called indecomposable if it has only trivial sum decomposition.

(ii) Let $a \in K^*$. Then the map

$$\langle a \rangle_r : K^r \to K \quad , \quad k_1, \ldots, k_r \mapsto a k_1 \cdots k_r$$

defines an indecomposable r-form $(K, \langle a \rangle_r)$ of dimension 1 over K.

(iii) A sum of 1-dimensional r-forms is called diagonal. We write

$$(K^n, \langle a_1, \ldots, a_n \rangle_r) := (K, \langle a_1 \rangle_r) \oplus \cdots \oplus (K, \langle a_n \rangle_r)$$

for an n-dimensional diagonal r-form.

(iv) Let L/K be a separable K-algebra of finite type, let $tr_{L/K} : L \to K$ be the trace map, and let $b \in L^*$. As in Definition 1.3(iv), we may see L as a K-vector space, and the map

$$tr_{L/K}\langle b \rangle_r : L^r \to K \quad , \quad l_1, \ldots, l_r \mapsto tr_{L/K}(b l_1 \cdots l_r)$$

defines an r-form $(L, tr_{L/K}\langle b \rangle_r)$ over K. This type of r-forms will frequently occur in the following sections.

1.5 Lemma. (Unique Decomposition of r-Forms) *Let (V, Θ) be an r-form over K.*

(i) There is a regular r-form (V', Θ') over K, unique up to isomorphism, such that (V, Θ) is isomorphic to the direct sum of (V', Θ') and a zero r-form.

(ii) Let $r \geq 3$ and let (V, Θ) be regular. Then (V, Θ) is isomorphic to a direct sum of indecomposable regular r-forms, which are uniquely defined up to isomorphism and order.

Proof: (i) One checks that $V_0 := \{v \in V \mid \Theta_{(v)} = 0\} \subset V$ is a K-vector subspace. Choose a complement $V' \subset V$ such that $V = V' \oplus V_0$ and let $\Theta' := \Theta|_{V'}$.
(ii) Existence is clear. In order to prove its uniqueness, it suffices to show that the intersection of two direct summands is again a direct summand.
We prove a preparational argument: For any subspace $U \subset V$, let

$$U^\perp := \{v \in V \mid \Theta_{(u,v)} = 0 \text{ for all } u \in U\} \subset V.$$

We claim that a subspace U in a regular space V is a direct summand in V if $U + U^\perp = V$. In fact, V is regular if and only if $V^\perp = 0$, and then $U + U^\perp = V$ implies $U \cap U^\perp \subset U^{\perp\perp} \cap U^\perp = (U + U^\perp)^\perp = V^\perp = 0$, which means that U is a direct summand.
Now let $U, W \subset V$ be direct summands. We claim that $U = U \cap W + U \cap W^\perp$. Since $U \cap W^\perp \subset U \cap (U \cap W)^\perp$, this implies $U = U \cap W + U \cap (U \cap W)^\perp$, and by the previous argument this shows that $U \cap W$ is a direct summand in U, and thus also in V, which finishes the proof.
Let $u \in U$ and let $u' \in W, u'' \in W^\perp$ such that $u = u' + u''$. Let $\tilde{u} = \tilde{u}' + \tilde{u}'' \in U^\perp$ and let $v_3 = v_3' + v_3'', \dots, v_r = v_r' + v_r'' \in V$ with $\tilde{u}', v_3', \dots, v_r' \in W$, $\tilde{u}'', v_3'', \dots, v_r'' \in W^\perp$. Then

$$\begin{aligned}\Theta(u', \tilde{u}, v_3, \dots, v_r) &= \Theta(u', \tilde{u}' + \tilde{u}'', v_3' + v_3'', \dots, v_r' + v_r'') = \Theta(u', \tilde{u}', v_3', \dots, v_r') \\ &= \Theta(u' + u'', \tilde{u}' + \tilde{u}'', v_3', \dots, v_r') = \Theta(u, \tilde{u}, v_3', \dots, v_r') = 0\,.\end{aligned}$$

This shows that $\Theta_{(u',\tilde{u})} = 0$ for all $\tilde{u} \in U^\perp$, so that we have $u' \in U^{\perp\perp} = U$. Hence $u = u' + u'' \in U \cap W + U \cap W^\perp$, which proves the Lemma. □

1.6 Notation. *The uniqueness statement in Lemma 1.5(i) allows us to introduce the following notation: In what follows, an r-form shall denote the isomorphism class of a regular r-form.*

1.7 Example. *The following example shows that the statement of Lemma 1.5(ii) is wrong for $r = 2$: Let K be a finite field of odd characteristic and let $a \in K^*$ such that $\bar{a} \neq 1$ in K^*/K^{*2}. Then $\langle 1, 1\rangle_2 = \langle a, a\rangle_2$, but $\langle 1\rangle_2 \neq \langle a\rangle_2$ ([28], Chap. 2, Th. 3.8).*

1.8 Theorem. (Witt Cancellation) *Let $r \geq 2$ and let U, V and W be r-forms. Then $U \oplus W \cong V \oplus W$ if and only if $U \cong V$.*

Proof: For $r \geq 3$, this follows from Lemma 1.5. For $r = 2$, this is classical Witt Cancellation (cf. [28], Chap. 1, Cor. 5.8). □

1.9 Theorem. (The Ring of r-Forms)

(i) Let $\hat{W}_r^+(K)$ denote the set of isomorphism classes of regular r-forms over K. Then $\hat{W}_r^+(K)$, together with the direct sum and the tensor product of r-forms given in Definition 1.3, forms a commutative semiring with unit element $(K, \langle 1\rangle_r)$.

(ii) Let $\hat{W}_r(K)$ denote the Grothendieck ring associated to $\hat{W}_r^+(K)$ (cf. [28], Chap. 2, Th. 1.1). Then $\hat{W}_r(K)$ is a commutative K-algebra with unit element $(K, \langle 1\rangle_r)$, we call it $\hat{W}_r(K)$ the Witt-Grothendieck ring of r-forms over K. The elements of $\hat{W}_r(K)$ are equivalence classes of formal differences $[\Theta - \Psi]$ of regular r-forms Θ, Ψ over K, where $\Theta - \Psi$ and $\Theta' - \Psi'$ are equivalent if there is an isomorphism of r-forms $\Theta \oplus \Psi' \cong \Theta' \oplus \Psi$. The map $\Theta \mapsto [\Theta - 0]$ gives a canonical embedding of semirings $\hat{W}_r^+(K) \subset \hat{W}_r(K)$.

(iii) Let $r \geq 3$. Then the additive group of $\hat{W}_r(K)$ is a free abelian group, generated by the isomorphism classes of indecomposable r-forms over K.

(iv) The unit group in $\hat{W}_r(K)$ consists of the 1-dimensional r-forms and is isomorphic to the group K^*/K^{*r}.

(v) The diagonal r-forms generate a subring $\hat{W}_r^D(K) \subset \hat{W}_r(K)$, which is isomorphic to the group ring $\mathbb{Z}[K^*/K^{*r}]$.

(vi) Let L/K be a field extension. In Definition 1.3(iii), we defined the scalar extension of r-forms over K with L. This induces a ring homomorphism

$$\begin{array}{rcl} \hat{W}_r(K) & \to & \hat{W}_r(L), \\ (V,\Theta) & \mapsto & (V,\Theta)_L\,. \end{array}$$

(vii) Let L/K be a finite field extension and let $0 \neq t \in \mathrm{Hom}_K(L,K)$. By the construction in Definition 1.3(iv), t induces a morphism of $\hat{W}_r(K)$-modules

$$\begin{array}{rcrcl} t & : & \hat{W}_r(L) & \to & \hat{W}_r(K), \\ & & (V,\Theta) & \mapsto & t(V,\Theta)\,. \end{array}$$

Proof: (i) In order to prove the ring axioms, it is easily checked that the obvious isomorphisms of K-vector spaces are in fact isomorphisms of r-forms and that they are compatible with isomorphism classes.
(ii) The formulation of the equivalence relation and the injectivity of the canonical map $\hat{W}_r^+(K) \to \hat{W}_r(K)$ require the statement on Witt Cancellation proved in Theorem 1.8.
(iii) This is clear from Lemma 1.5.
(iv)-(vii) are clear. □

1.10 Lemma. (Scalar Extension and Tensor Product of Trace Forms) *Let L/K be an algebraic field extension and let M/K be a finite separable field extension, both contained in a separable closure $\bar{K}$. Let $\sigma_1, \ldots, \sigma_s \in \mathrm{Hom}_K(M, \bar{K})$ be representatives for the different orbits under G_L-action by left translation. Let (W, Ψ) be an r-form over M. For $\sigma \in \mathrm{Hom}_K(M, \bar{K})$, let $L\,\sigma M \subset \bar{K}$ denote the composite field containing L and σM and let $(W \otimes_{\sigma M} L\,\sigma M, \Psi_{L\,\sigma M})$ denote the r-form over $L\,\sigma M$ defined by $\Psi_{L\,\sigma M}(w_1 \otimes x_1, \ldots, w_r \otimes x_r) := \prod_k x_k \cdot \sigma\Psi(w_1, \ldots, w_r)$ for $w_k \in W$, $x_k \in L\,\sigma M$ (note that for $m \in M, w \in W, x \in L\,\sigma M$ we have $mw \otimes x = w \otimes \sigma m\, x$ in $W \otimes_{\sigma M} L\,\sigma M$, so that $(W \otimes_{\sigma M} L\,\sigma M, \Psi_{L\,\sigma M})$ is a well defined r-form over $L\,\sigma M$).*

(i) There is an isomorphism of K-algebras

$$\begin{array}{rcl} L \otimes_K M & \xrightarrow{\sim} & \bigoplus_{i=1}^{s} L\,\sigma_i M \\ l \otimes m & \mapsto & (l\,\sigma_i m)_i\,. \end{array}$$

The orbit decomposition of $\mathrm{Hom}_K(M,\bar{K})$ *under* G_L*-action is given by*

$$\mathrm{Hom}_K(M,\bar{K}) = \bigcup_{i=1}^{s} \mathrm{Hom}_L(L\,\sigma_i M,\bar{K})\,\sigma_i\,.$$

(ii) There is an isomorphism of r*-forms over* L

$$\begin{array}{rcl} (W\otimes_K L,(\mathrm{tr}_{M/K}\Psi)_L) & \xrightarrow{\sim} & \bigoplus_{i=1}^{s}\big(W\otimes_{\sigma_i M} L\,\sigma_i M,\mathrm{tr}_{L\sigma_i M/L}(\Psi_{L\,\sigma M})\big) \\ w\otimes_K l & \mapsto & (w\otimes_{\sigma_i M} l)_i \end{array}$$

In particular, for $c\in M^*$ *we have*

$$(M,tr_{M/K}\langle c\rangle_r)\otimes_K L\cong \bigoplus_{i=1}^{s}(L\,\sigma_i M,\mathrm{tr}_{L\sigma_i M/L}\langle \sigma_i c\rangle_r)\,.$$

(iii) Let L/K *be finite separable, and let* (V,Θ) *be an* r*-form over* L*. Then there is an isomorphism of* r*-forms over* K

$$\begin{array}{l} (V,tr_{L/K}\Theta)\otimes_K(W,tr_{M/K}\Psi)\xrightarrow{\sim} \\ \bigoplus_{i=1}^{s}\big((V\otimes_L L\,\sigma_i M)\otimes_{L\,\sigma_i M}(W\otimes_{\sigma_i M} L\,\sigma_i M),tr_{L\,\sigma_i M/K}(\Theta_{L\,\sigma_i M}\otimes\Psi_{L\,\sigma_i M})\big)\,, \\ v\otimes w\mapsto((v\otimes_L 1)\otimes_{L\,\sigma_i M}(w\otimes_{\sigma_i M}1))_{i=1,\dots,s} \end{array}$$

In particular, for $b\in L^*$ *and* $c\in M^*$ *we have*

$$(L,tr_{L/K}\langle b\rangle_r)\otimes_K(M,tr_{M/K}\langle c\rangle_r)\cong\bigoplus_{i=1}^{s}(L\,\sigma_i M,\mathrm{tr}_{L\sigma_i M/K}\langle b\,\sigma_i c\rangle_r)\,.$$

Proof: (i) Let $a\in M$ be a primitive element for M/K and let $f\in K[x]$ be its minimal polynomial. Then $f=\prod_\sigma(x-\sigma a)$, where the product runs over $\sigma\in\mathrm{Hom}_K(M,\bar{K})$ and the orbit decomposition under G_L-action gives the decomposition $f=f_1\cdots f_s$ to irreducible elements in $L[x]$. We have an isomorphism of K-algebras $L\otimes_K M\cong L\otimes_K K[x]/(f)\cong L[x]/(f)=L[x]/(f_1\cdots f_s)$, and since f is separable, the Chinese Remainder Theorem gives a decomposition $L[x]/(f_1\cdots f_s)\cong\bigoplus_i L[x]/(f_i)\cong\bigoplus_i L(\sigma_i a)\cong\bigoplus_i L\,\sigma_i M$. For $l\in L$ and $m\in M$, we have $l\otimes 1\mapsto(l,\dots,l)$, $1\otimes m\mapsto(\sigma_1 m,\dots,\sigma_s m)$ under this isomorphism. Hence $l\otimes m=(l\otimes 1)(1\otimes m)\mapsto(l\,\sigma_1 m,\dots,l\,\sigma_s m)$. We have seen that the set of linear factors of f_i is bijective both to the orbit of σ_i and to the set $\mathrm{Hom}_L(L\,\sigma_i M,\bar{K})$ and these bijections give the formula for the orbit decomposition.
(ii) Clearly our map is L-linear, and next we will show that it is a morphism of r-forms. From (i) we know $\mathrm{tr}_{M/K}=\sum_i \mathrm{tr}_{L\sigma_i M/L}\,\sigma_i$, hence for $w_1,\dots,w_r\in W$ and $l_1,\dots,l_r\in L$ we have

$$
\begin{aligned}
(\mathrm{tr}_{\mathrm{M/K}}\Psi)_L\,(w_1\otimes l_1,\dots,w_r\otimes l_r) &= \prod_i l_i\cdot \mathrm{tr}_{M/K}\Psi(w_1,\dots,w_r)\\
&= \prod_i l_i\cdot\sum_i \mathrm{tr}_{L\sigma_iM/L}(\sigma_i\Psi(w_1,\dots,w_r)) = \sum_i \mathrm{tr}_{L\sigma_iM/L}(\prod_i l_i\cdot\sigma_i\Psi(w_1,\dots,w_r))\\
&= \sum_i \mathrm{tr}_{L\sigma_iM/L}(\Psi_{L\,\sigma_iM})(w_1\otimes l_1,\dots,w_r\otimes l_r)\,.
\end{aligned}
$$

It remains to show that our map is bijective. It is injective, since it is a morphism of regular r-forms and from (i) we know that $[M:K]=\sum_i [L\sigma_iM:L]$. Hence

$$
\begin{aligned}
\dim_L(W\otimes_K L) &= \dim_K(W) = \dim_M(W)\cdot[M:K] = \dim_M(W)\cdot\sum_i [L\sigma_iM:L]\\
&= \sum_i \dim_{L\,\sigma_iM}(W\otimes_{\sigma_iM} L\,\sigma_iM)\cdot[L\sigma_iM:L] = \dim_L(\bigoplus_i W\otimes_{\sigma_iM} L\,\sigma_iM),
\end{aligned}
$$

which proves that it is surjective.

(iii) We proceed as in (ii): Our map is K-linear, and since $\mathrm{tr}_{\mathrm{M/K}}=\sum_i \mathrm{tr}_{L\sigma_iM/L}\,\sigma_i$, we have

$$
\begin{aligned}
\mathrm{tr}_{\mathrm{L/K}}(l)\cdot\mathrm{tr}_{\mathrm{M/K}}(m) &= \mathrm{tr}_{\mathrm{L/K}}(l\cdot\mathrm{tr}_{\mathrm{M/K}}(m))\\
&= \mathrm{tr}_{\mathrm{L/K}}(l\cdot\sum_i \mathrm{tr}_{L\sigma_iM/L}(\sigma_i m)) = \sum_i \mathrm{tr}_{L\sigma_iM/K}(l\,\sigma_i m)
\end{aligned}
$$

for $l\in L$ and $m\in M$. Then for $v_1,\dots,v_r\in V$, $w_1,\dots,w_r\in W$ we have

$$
\begin{aligned}
&(\mathrm{tr}_{\mathrm{L/K}}\Theta\otimes\mathrm{tr}_{\mathrm{M/K}}\Psi)(v_1\otimes w_1,\dots,w_1\otimes w_r)\\
&= \mathrm{tr}_{\mathrm{L/K}}\Theta(v_1,\dots,v_r)\cdot\mathrm{tr}_{\mathrm{M/K}}\Psi(w_1,\dots,w_r)\\
&= \sum_i \mathrm{tr}_{L\sigma_iM/K}(\Theta(v_1,\dots,v_r)\cdot\sigma_i\Psi(w_1,\dots,w_r))\\
&= \sum_i \mathrm{tr}_{L\sigma_iM/K}(\Theta_{L\,\sigma_iM}(v_1\otimes 1,\dots,v_r\otimes 1)\cdot\Psi_{L\,\sigma_iM}(w_1\otimes 1,\dots,w_r\otimes 1))\\
&= \sum_i \mathrm{tr}_{L\sigma_iM/K}\Theta_{L\,\sigma_iM}\otimes\Psi_{L\,\sigma_iM}((v_1\otimes 1)\otimes(w_1\otimes 1),\dots,(v_r\otimes 1)\otimes(w_r\otimes 1))\,.
\end{aligned}
$$

This shows that our map is a morphism of r-forms. Bijectivity follows with the same argument as in (ii). □

2 Multilinear and Homogeneous r-forms

Over a field of characteristic $\neq 2$, bilinear forms correspond to quadratic forms, i.e. to homogeneous polynomials of degree 2. In this section, we will examine the analogous correspondence between symmetric multilinear r-forms and homogeneous polynomial of degree r. Keep in mind the previous assumption that $r!$ is invertible in K.

The following notation will be used for writing homogeneous polynomials of degree r in n variables. For $n \in \mathbb{N}$, let $I(r,n) \subset \mathbb{N}^n$ be the set of non-negative n-tuples $\nu = (\nu_1, \ldots, \nu_n)$ such that $\nu_1 + \cdots + \nu_n = r$. For $\nu \in I(r,n)$ we write $\binom{r}{\nu} := \frac{r!}{\nu_1! \cdots \nu_n!}$ and $x^\nu := x_1^{\nu_1} \cdots x_n^{\nu_n} \in K[x_1, \ldots, x_n]$. Using this notation, any homogeneous polynomial of degree r in n variables over K has the form $f(x_1, \ldots, x_n) = \sum\limits_{\nu \in I(r,n)} a_\nu x^\nu$ with coefficients $a_\nu \in K(\nu \in I(r,n))$. We say that $f, g \in K[x_1, \ldots, x_n]$ are isomorphic if there is $\varphi \in \mathrm{GL}_n(K)$ such that $f(x) = g(\varphi x)$.

2.1 Lemma. (Multilinear and Homogeneous r-Forms)
Let V be a K-vector space and let $\{v_1, \ldots, v_n\}$ be a K-basis of V. For an r-form (V, Θ) on V, let $f = f_\Theta \in K[x_1, \ldots, x_n]$ be defined by

$$f(x_1, \ldots, x_n) := \Theta(\Sigma_i x_i v_i, \ldots, \Sigma_i x_i v_i).$$

Then f is a homogeneous polynomial of degree r and the map $\Theta \mapsto f_\Theta$ gives a bijection between isomorphism classes of r-forms over K and isomorphisms of homogeneous polynomials of degree r in n variables. In particular, an r-form (V, Θ) is determined by its values $\Theta(v, \ldots, v)$ for $v \in V$.

Proof: Let $f \in K[x_1, \ldots, x_n]$ be homogeneous of degree r. Since $r!$ is invertible in K, we can write $f(x) = \sum_{\nu \in I(r,n)} \binom{r}{\nu} a_\nu x^\nu$ with $a_\nu \in K$. Now there is a unique r-form Θ on V satisfying $\Theta(\underbrace{v_1, \ldots, v_1}_{\nu_1}, \ldots, \underbrace{v_n, \ldots, v_n}_{\nu_n}) := a_\nu$ for $\nu \in I(r,n)$ and one checks $f = f_\Theta$. Clearly, this correspondence respects isomorphism classes. $\square$

With the identification from the Lemma, we may switch between the viewpoints of multilinear r-forms and homogeneous polynomials of degree r over K, which we shall call homogenous r-forms for convenience. We shall simply speak of r-forms, if there is no ambiguity.

Let (V, Θ) be a multilinear r-form, let $\{v_i\}$ be a basis, and let the homogeneous r-form f be given as in the Lemma. We denote this correspondence by $\Theta \overset{\{v_1, \ldots, v_n\}}{\longleftrightarrow} f$. We say that the multilinear r-form (V, Θ) is non-singular if the corresponding homogeneous r-form f is non-singular, meaning that the projective hypersurface described by f in non-singular.

2.2 Lemma. (Homogeneous r-forms and k-Regularity)
Let (V, Θ) be an r-form over K, let $\{v_1, \dots, v_n\}$ be a K-basis of V and let $f \in K[x_1, \dots, x_n]$ such that $\Theta \xleftrightarrow{\{v_i\}} f$.

(i) Let $1 \le k < r$. Then $\frac{r!}{(r-k)!} \cdot \Theta_{(v_{i_1}, \dots, v_{i_k})} \xleftrightarrow{\{v_i\}} \frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}}(f)$ for $1 \le i_j \le n$.

(ii) We say that $x \in K^n$ is a k-fold zero of f if all k-fold partial derivatives of f vanish at x. (V, Θ) is $(r-k)$-regular if and only if the only k-fold zero of f is $x = 0$. In particular, Θ is non-singular if and only if it is $(r-1)$-regular over the separable closure $\bar{K}$.

Proof: (i) For $x_1, \dots, x_n, y \in K$, the correspondence $\Theta \xleftrightarrow{\{v_i\}} f$ gives

$$f(x_1, \dots, x_j + y, \dots, x_n) - f(x_1, \dots, x_n) = y \cdot r\Theta(v_j, \Sigma_i x_i v_i, \dots, \Sigma_i x_i v_i) + y^2(\dots).$$

Hence the differential quotient yields $\frac{\partial}{\partial x_j}(f) \xleftrightarrow{\{v_i\}} r \cdot \Theta_{(v_j)}$. This proves (i) for $k = 1$ and one proceeds by induction on k.
(ii) By (i), a k-fold zero x of f corresponds to a vector $v = \Sigma_i x_i v_i \in V$ with $\Theta_{(v, \dots, v)} = 0$, where v occurs $r - k$ times, and vice versa. □

2.3 Lemma. (Homogeneous r-Forms and Tensor Product)
Let (V, Θ) and (W, Ψ) be r-forms over K with bases $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$. Let $I(r, n \times m)$ be the set of $(n \times m)$-tuples of weight r. For $\lambda = (\lambda_{ij}) \in I(r, n \times m)$, let $\Sigma_j \lambda_{\bullet j} \in I(r, n)$ and $\Sigma_i \lambda_{i \bullet} \in I(r, m)$ denote the sum of columns (rows respectively) of λ. Let $f \in K[x_1, \dots, x_n]$ such that $\Theta \xleftrightarrow{\{v_i\}} f$, let $g \in K[x_1, \dots, x_m]$ such that $\Psi \xleftrightarrow{\{w_j\}} g$, and let $f \otimes g \in K[x_{11}, \dots, x_{nm}]$ such that $\Theta \otimes \Psi \xleftrightarrow{\{v_i \otimes w_j\}} f \otimes g$. Assume that $f(x_1, \dots, x_n) = \sum\limits_{\nu \in I(r,n)} \binom{r}{\nu} a_\nu x^\nu$ and that $g(x_1, \dots, x_m) = \sum\limits_{\mu \in I(r,m)} \binom{r}{\mu} b_\mu x^\mu$ with coefficients $a_\nu, b_\mu \in K$. Then

$$(f \otimes g)(x_{ij}) = \sum_{\lambda \in I(r, n \times m)} \binom{r}{\lambda} a_{(\Sigma_j \lambda_{\bullet j})} b_{(\Sigma_i \lambda_{i \bullet})} x^\lambda.$$

Proof: In Lemma 2.1, we saw that the coefficients of f and g are given by

$$a_\nu = \Theta(\underbrace{v_1, \dots, v_1}_{\nu_1}, \dots, \underbrace{v_n, \dots, v_n}_{\nu_n}) \quad , \quad b_\mu = \Psi(\underbrace{w_1, \dots, w_1}_{\mu_1} \dots, \underbrace{w_m, \dots, w_m}_{\mu_m})$$

for $\nu \in I(r, n)$ and $\mu \in I(r, m)$. Writing $f \otimes g = \Sigma_{\lambda \in I(r, n \times m)} \binom{r}{\lambda} c_\lambda x^\lambda$ we have

$$\begin{aligned} c_\lambda &= \Theta \otimes \Psi(\underbrace{v_1 \otimes w_1, \dots, v_1 \otimes w_1}_{\lambda_{11}}, \dots, \underbrace{v_n \otimes w_m, \dots, v_n \otimes w_m}_{\lambda_{nm}}) \\ &= \Theta(\underbrace{v_1, \dots, v_1}_{\Sigma_j \lambda_{1j}}, \dots, \underbrace{v_n, \dots, v_n}_{\Sigma_j \lambda_{nj}}) \cdot \Psi(\underbrace{w_1, \dots, w_1}_{\Sigma_i \lambda_{i1}}, \dots, \underbrace{w_m, \dots, w_m}_{\Sigma_i \lambda_{im}}) \\ &= a_{(\Sigma_j \lambda_{\bullet j})} b_{(\Sigma_i \lambda_{i \bullet})}. \end{aligned}$$

□

2.4 Lemma. (Tensor Product and k-Regularity)
Let (V,Θ), (W,Ψ) be r-forms over K.

(i) Let $1 \le k \le r-1$. If $\Theta \otimes_K \Psi$ is k-regular, then both Θ and Ψ are k-regular.

(ii) If Θ and Ψ are regular, then $\Theta \otimes_K \Psi$ is regular.

(iii) For $k > 1$, the tensor product of k-regular r-forms is not k-regular in general.

Proof: (i) Let $0 \ne v \in V$ with $\Theta_{(v,\dots,v)} = 0$. Then $(\Theta \otimes \Psi)_{(v\otimes w,\dots,v\otimes w)} = 0$ for all $w \in W$.
(ii) Assume that Ψ is regular, and that $\Theta\otimes\Psi$ is not. Let $\{v_i\}$ and $\{w_j\}$ be K-bases for V and W and choose $0 \ne u = \sum_{i,j} u_{ij}\, v_i \otimes w_j$ such that $(\Theta \otimes \Psi)_{(u)} = 0$. Then for all $i_2, j_2, \dots, i_r, j_r$ we have

$$0 = \Theta \otimes \Psi(u, v_{i_2} \otimes w_{j_2}, \dots, v_{i_r} \otimes w_{j_r}) = \sum_{i,j} u_{ij}\Theta(v_i, v_{i_2}, \dots, v_{i_r})\Psi(w_j, w_{j_2}, \dots, w_{j_r}).$$

For $i = 1,\dots,n$, let $u_i := \sum_j u_{ij} w_j \in W$, i.e. $u = \sum_i v_i \otimes u_i$. Choose k such that $u_k \ne 0$. Since Ψ is regular, there exist $j_2,\dots,j_r$ such that $0 \ne \Psi(u_k, w_{j_2}, \dots, w_{j_r}) = \sum_j u_{kj}\Psi(w_j, w_{j_2}, \dots, w_{j_r})$. For $i = 1,\dots,n$, let $y_i := \Psi(u_i, w_{j_2}, \dots, w_{j_r}) \in K$ and let $y = y(j_2,\dots,j_r) := \sum_i y_i v_i \in V$. Since $y_k \ne 0$, we have $y \ne 0$. But for all $i_2,\dots,i_r$ we have

$$\begin{aligned}\Theta(y, v_{i_2}, \dots, v_{i_r}) &= \sum_i y_i \Psi(v_i, v_{i_2}, \dots, v_{i_r})\\ &= \sum_{i,j} u_{ij}\Theta(v_i, v_{i_2}, \dots, v_{i_r})\Psi(w_j, w_{j_2}, \dots, w_{j_r}) = 0.\end{aligned}$$

This shows that Θ is not regular.
(iii) We provide an example: The polynomials $x_1^r + x_1x_2^{r-1} \pm x_1x_3^{r-1}$ are non-singular. Using the notation of the previous Lemma with the standard basis in K^2, one checks that

$$\begin{aligned}&(x_1^r + x_1x_2^{r-1} + (-1)^r x_1x_3^{r-1}) \otimes (x_1^r + x_1x_2^{r-1} + x_1x_3^{r-1})\\ &= x_{11}^r + x_{11}x_{12}^{r-1} + x_{11}x_{13}^{r-1} + x_{11}x_{21}^{r-1} + (-1)^r x_{11}x_{31}^{r-1}\\ &+\tfrac{1}{r}(x_{11}x_{22}^{r-1} + x_{11}x_{23}^{r-1} + (-1)^r x_{11}x_{32}^{r-1} + (-1)^r x_{11}x_{33}^{r-1})\\ &+\tfrac{r-1}{r}(x_{12}x_{21}x_{22}^{r-1} + x_{12}x_{31}x_{32}^{r-1} + (-1)^r x_{13}x_{21}x_{23}^{r-1} + (-1)^r x_{13}x_{31}x_{33}^{r-1}).\end{aligned}$$

This r-form has a singularity at $x_{11} = x_{12} = x_{13} = x_{21} = x_{22} = x_{31} = x_{33} = 0$, $x_{23} = 1, x_{32} = -1$. □

3 The Center of r-Forms, Separable r-Forms

3.1 Definition. (The Center of an r-Form)
Let $r \geq 3$ and let (V, Θ) be an r-form over K. We define

$$\mathrm{Cent}(\Theta) = \mathrm{Cent}_K(V, \Theta) := \{\varphi \in \mathrm{End}_K(V) \mid \Theta_{(\varphi u, v)} = \Theta_{(u, \varphi v)} \text{ for all } u, v \in V\}.$$

3.2 Lemma. *Let $r \geq 3$ and let (V, Θ) and (W, Ψ) be r-forms over K.*

(i) $\mathrm{Cent}(\Theta)$ *is a commutative K-algebra.*

(ii) The r-form (V, Θ) is indecomposable if and only if $\mathrm{Cent}(\Theta)$ *is an irreducible algebra.*

(iii) $\mathrm{Cent}(\Theta \otimes_K \Psi) = \mathrm{Cent}(\Theta) \otimes_K \mathrm{Cent}(\Psi)$.

(iv) Let L/K be a field extension. Then $\mathrm{Cent}_L(V \otimes_K L, \Theta_L) = \mathrm{Cent}_K(V, \Theta) \otimes_K L$.

(v) Let L/K be a finite field extension, let (U, Φ) be an r-form over L and let $0 \neq t \in \mathrm{Hom}_K(L, K)$. *Then* $\mathrm{Cent}_K(U, t\Phi) = \mathrm{Cent}_L(U, \Phi)$.

(vi) Let $\{v_1, \ldots, v_n\}$ be a K-basis of V and let $f \in K[x_1, \ldots, x_n]$ be given by $\Theta \xleftrightarrow{\{v_i\}} f$. Then $\mathrm{Cent}(\Theta) \cong \{M \in M_n(K) \mid M^t J = JM\}$, *where $J = (\frac{\partial^2 f}{\partial x_i \partial x_j})$ is the Hesse matrix for f.*

Proof: (i) We need to show that multiplication in $\mathrm{Cent}(\Theta)$ is commutative and that $\mathrm{Cent}(\Theta)$ is closed under multiplication. Let $\varphi, \tau \in \mathrm{Cent}(\Theta)$, and let $u, v, w \in V$. Then $\Theta_{(\varphi\tau u, v, w)} = \Theta_{(\tau u, \varphi v, w)} = \Theta_{(u, \varphi v, \tau w)} = \Theta_{(\varphi u, v, \tau w)} = \Theta_{(\tau\varphi u, v, w)}$. Hence $\varphi\tau = \tau\varphi$, since Θ is regular. Now $\Theta_{(\varphi\tau u, v)} = \Theta_{(\tau u, \varphi v)} = \Theta_{(u, \tau\varphi v)} = \Theta_{(u, \varphi\tau v)}$, so that $\varphi\tau \in \mathrm{Cent}(\Theta)$.
(ii) A decomposition of r-forms clearly induces a decomposition of the center. Hence, let $\mathrm{Cent}(\Theta)$ be decomposable. Then there is a non-trivial idempotent element $f \in \mathrm{Cent}(\Theta)$ and one checks that $V = \mathrm{im} f \oplus \ker f$ is a direct sum of r-forms and (V, Θ) is decomposable.
(iii) The inclusion "$\supset$" is clear, hence we need to show "$\subset$". Choose a K-basis $f_1, \ldots, f_s$ of $\mathrm{Cent}_K(\Theta)$ ($g_1, \ldots, g_t$ of $\mathrm{Cent}(\Psi)$) and extend it to a K-basis $f_1, \ldots, f_n$ of $\mathrm{End}_K(V)$ ($g_1, \ldots, g_m$ of $\mathrm{End}_K(W)$).
Let $k \in \mathrm{Cent}_K(\Theta \otimes \Psi)$ Then k has the form $k = \sum_{i,j} a_{ij} f_i \otimes g_j$ with $a_{ij} \in K$ ($1 \leq i \leq n, 1 \leq j \leq m$). For $v_1, \ldots, v_r \in V$, let

$$h_1 = h_1(v_1, \ldots, v_r) := \textstyle\sum_{i,j} a_{ij}\, \Theta(f_i(v_1), v_2, \ldots, v_r)\, g_j \in \mathrm{End}(W),$$

$$h_2 = h_2(v_1, \ldots, v_r) := \textstyle\sum_{i,j} a_{ij}\, \Theta(v_1, f_i(v_2), v_3, \ldots, v_r)\, g_j \in \mathrm{End}(W).$$

Then for $w_1, \ldots, w_r \in W$ we have

$$\begin{aligned}
&\Psi(h_1(w_1), w_2, \ldots, w_r) \\
&\quad = \textstyle\sum_{i,j} a_{ij} \Theta(f_i(v_1), v_2, \ldots, v_r) \Psi(g_j(w_1), w_2, \ldots, w_r) \\
&\quad = \Theta \otimes \Psi(k(v_1 \otimes w_1), v_2 \otimes w_2, \ldots, v_r \otimes w_r) \\
&\quad = \Theta \otimes \Psi(v_1 \otimes w_1, k(v_2 \otimes w_2), v_3 \otimes w_3, \ldots, v_r \otimes w_r) \\
&\quad = \Psi(w_1, h_2(w_2), w_3, \ldots, w_r).
\end{aligned}$$

In other words, $\Psi_{(h_1 w_1, w_2)} = \Psi_{(w_1, h_2 w_2)}$. Thus,

$$\Psi_{(h_1w_1,w_2,w_3)} = \Psi_{(h_1w_1,w_3,w_2)} = \Psi_{(w_1,h_2w_3,w_2)} = \Psi_{(w_2,h_2w_3,w_1)} = \Psi_{(h_1w_2,w_3,w_1)} = \Psi_{(w_1,h_1w_2,w_3)},$$

and therefore $h_1 \in \mathrm{Cent}(\Psi)$. Now $h_1 = \Sigma_j \Theta(\Sigma_i a_{ij} f_i(v1), v_2, \dots, v_r)\, g_j$ and by the choice of the basis $\{g_j\}$ we have $\Theta(\Sigma_i a_{ij} f_i(v1), v_2, \dots, v_r) = 0$ for $j > t$ and $v_1, \dots, v_r \in V$. Since Θ is regular, this implies $\Sigma_i a_{ij} f_i = 0$ for $j > t$ and therefore $a_{ij} = 0$ for $j > t$. Similarly one shows $a_{ij} = 0$ for $i > s$, which finishes the proof.
(iv) Again, "$\supset$" is clear, and we need to show "$\subset$". Let $k \in \mathrm{Cent}_L(\Theta_L)$. Then k has the form $k = \sum_{i=1}^n f_i \otimes s_i$ with $f_i \in \mathrm{End}_K(V), s_i \in L$. Let $l_1, \dots, l_m$ be a K-basis for the K-submodule in L generated by $s_1, \dots, s_n$, and let $k = \sum_{i,j} a_{ij}(f_i \otimes l_j)$ with $a_{ij} \in K$. For $j = 1, \dots, m$ let $k_j := \sum_i a_{ij} f_i \in \mathrm{End}(V)$, so that $k = \sum_j k_j \otimes l_j$. For $v_1, \dots, v_r \in V$, we have

$$\begin{aligned}\sum_j l_j\, \Theta(k_j(v_1), v_2, \dots, v_r) &= \Theta_L(k(v_1), v_2, \dots, v_r)\\ &= \Theta_L(v_1, k(v_2), v_3 \dots, v_r) \sum_j l_j\, \Theta(v_1, k_j(v_2), v_3, \dots, v_r)).\end{aligned}$$

For $j = 1, \dots, m$, this implies $\Theta(k_j(v_1), v_2, \dots, v_r) = \Theta(v_1, k_j(v_2), v_3, \dots, v_r)$ by comparing the coefficient for l_j. Hence $k_j \in \mathrm{Cent}(\Theta)$ for $j = 1, \dots, m$, and therefore $k = \sum_j k_j \otimes l_j \in \mathrm{Cent}(\Theta) \otimes_K L$.
(v) Again, "$\supset$" is clear, and we need to show "$\subset$". Let $f \in \mathrm{Cent}_K(t\,\Psi) \subset \mathrm{End}_K(V)$. First we will show that f is L-linear. For $v_1, \dots, v_r \in V$ and $b, c \in L$ we have

$$\begin{aligned}&t(c\,\Psi(f(bv_1), v_2, \dots, v_r)) = t\Psi(f(bv_1), v_2, \dots, cv_r) = t\Psi(bv_1, f(v_2), \dots, cv_r)\\ &= t\Psi(v_1, f(v_2), \dots, bcv_r) = t\Psi(f(v_1), v_2, \dots, bcv_r) = t(c\,\Psi(bf(v_1), v_2, \dots, v_r)),\end{aligned}$$

and therefore $\Theta(f(bv_1), v_2, \dots, v_r) = \Theta(bf(v_1), v_2, \dots, v_r)$ since t is non-trivial. Since Ψ is regular, this implies that f is L-linear and a similar argument shows $f \in \mathrm{Cent}_L(\Psi)$.
(vi) Let $\varphi \in \mathrm{End}_K(V)$, and let $M \in M_n(K)$ represent φ in the basis $\{v_i\}$. By Lemma 2.2(i), we have $\Theta_{(\varphi v_i, v_j)} = \sum_\nu M_{\nu i} \Theta_{(v_\nu, v_j)} \xleftrightarrow{\{v_i\}} \sum_\nu M_{\nu i} J_{j\nu} = (JM)_{ij}$ and analogously $\Theta_{(v_i, \varphi v_j)} \xleftrightarrow{\{v_i\}} (M^t J)_{ij}$. Hence $\varphi \in \mathrm{Cent}(\Theta)$ if and only if $M^t J = JM$. □

Remark. For $r = 2$, Definition 3.1 does not give a K-algebra: Let $x \in K$, $x \neq 1, 0$, and consider the quadratic form $\langle 1, x \rangle_2$. Then $\left(\begin{smallmatrix}1 & 0\\ 0 & x\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0 & x\\ 1 & 0\end{smallmatrix}\right)$ are in the center, but their product is not.

Example. Let $r \geq 3$, and consider the r-form $x^r + x^{r-1}y$. Using Lemma 3.2, one checks that its center is isomorphic to the irreducible K-algebra $K[x]/(x^2)$. In particular, this shows that indecomposable r-forms of dimension > 1 exist over any field, in contrast to the situation in degree 2.

The following Lemma is cited from ([24], Chap. 1, Prop. 3.1):

3.3 Lemma. *For a finite-dimensional commutative K-algebra A, the following conditions are equivalent:*

(i) The Jacobson radical of $A \otimes_K \bar{K}$ is zero.

(ii) $A \otimes_K \bar{K}$ is isomorphic to a finite product of copies of $\bar{K}$.

(iii) A is isomorphic to a finite product of separable field extensions of K.

(iv) The trace pairing $A \times A \to K, (a, b) \mapsto tr_{A/K}(ab)$ is non-degenerate. □

3.4 Definition. *A K-algebra is called separable if it satisfies the conditions in Lemma 3.3.*

The following Lemma is a collection of statements from [8]:

3.5 Lemma. *Let A and B be K-algebras and let L be a field extension of K. Let $A \otimes_K B$ denote the K-algebra with product given by $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$.*

(i) $A \otimes_K B$ is separable over K if and only if A and B are separable over K.

(ii) $A \otimes_K L$ is separable over L if and only if A is separable over K

(iii) Let A be an L-algebra which is separable over K. Then A is separable over L.

Proof: (i) follows from ([8], Chap. 2, Prop. 1.6 and Cor. 1.9).
(ii) follows from (loc. cit., Corollaries 1.7 and 1.10).
(iii) is found in (loc. cit., Prop. 1.12). □

3.6 Lemma.

(i) Let (V, Θ) be 2-regular over K. Then $\mathrm{Cent}(V, \Theta)$ is a reduced ring. Furthermore, if V is 2-regular over $\bar{K}$, then $\mathrm{Cent}(\Theta)$ is a separable K-algebra.

(ii) If $L := \mathrm{Cent}(V, \Theta)$ is a reduced ring, then $[L : K] \leq \dim_K(V)$.

(iii) If $L := \mathrm{Cent}(V, \Theta)$ be a field, such that $[L : K] \geq \dim_K(V)$. Then there is an isomorphism of r-forms $(V, \Theta) \cong (L, t\langle 1 \rangle_r)$ for a non-zero $t \in \mathrm{Hom}_K(L, K)$. If L/K is separable, then $(V, \Theta) \cong (L, tr_{L/K}\langle b \rangle_r)$ for some $b \in L^*$.

Proof: (i) Let $0 \neq t \in \mathrm{Cent}(\Theta)$ such that $t^2 = 0$, and choose $v \in V$ with $t(v) \neq 0$. Then $\Theta_{(tv,tv)} = \Theta_{(t^2 v,v)} = 0$, in contradiction to the assumption that Θ is 2-regular. This shows that $\mathrm{Cent}(\Theta)$ has zero nil radical. If V is 2-regular over $\bar{K}$, then the same argument shows that $\mathrm{Cent}(\Theta) \otimes \bar{K}$ has zero nil radical, hence it is separable by Lemma 3.3.

(ii) L is an artinian reduced K-algebra, hence it is isomorphic to a finite product of finite field extensions of K. By Lemma 3.2(ii), we may therefore assume that L is a field. V is an L-vector space, and counting dimensions one checks that $\dim_K(V) = [L:K] \cdot \dim_L(V) \geq [L:K]$.
(iii) By (ii), we have $[L:K] = \dim_K(V)$. Choose $0 \neq v \in V$ and define maps $f : L \to V, f(l) := lv$ and $t : L \to K, t(l) := \Theta(lv, v, \ldots, v)$. Clearly t is K-linear, f is an isomorphism of K-vector spaces and for $l_1, \ldots, l_r \in L$ we have

$$\Theta(f(l_1), \ldots, f(l_r)) = \Theta(l_1 v, \ldots, l_r v) = \Theta(l_1 \cdots l_r v, v, \ldots, v) = t(l_1 \cdots l_r).$$

Hence $f : (L, t\langle 1\rangle_r) \to (V, \Theta)$ is an isomorphism of r-forms over K. Now let L be separable. By Lemma 3.3(iv), the trace pairing induces an isomorphism of K-vector spaces $L \xrightarrow{\sim} \mathrm{Hom}_K(L, K)$. Hence we have $t\langle 1\rangle_r = \mathrm{tr}_{L/K}\langle b\rangle_r$ for some $b \in L^*$. □

3.7 Definition. *An r-form (V, Θ) over K is called separable if $\mathrm{Cent}_K(V)$ is a separable K-algebra and $\dim_K(V) \leq \dim_K(\mathrm{Cent}_K(V))$.*

3.8 Lemma. (Separable r-Forms)
Let $r \geq 3$ and let (V, Θ) and (W, Ψ) be r-forms over K.

(i) If (V, Θ) is separable, then $\dim_K(V) = \dim_K(\mathrm{Cent}_K(V))$.

(ii) Let (V, Θ) be indecomposable. Then (V, Θ) is separable over K if and only if there is a finite separable field extension L/K and $b \in L^*$ such that $(V, \Theta) \xrightarrow{\sim} (L, tr_{L/K}\langle b\rangle_r)$. Two indecomposable separable r-forms $(L, tr_{L/K}\langle b\rangle_r)$ and $(M, tr_{M/K}\langle c\rangle_r)$ over K are isomorphic if and only if there is a K-linear isomorphism of fields $\varphi : L \xrightarrow{\sim} M$ such that $\varphi(b) \equiv c \mod M^{*r}$.

(iii) $(V, \Theta) \oplus (W, \Psi)$ is separable if and only if (V, Θ) and (W, Ψ) are separable.

(iv) $(V, \Theta) \otimes (W, \Psi)$ is separable if and only if (V, Θ) and (W, Ψ) are separable.

(v) Let L/K be a field extension. (V, Θ) is separable over K if and only if $(V \otimes_K L, \Theta_L)$ is separable over L. In particular, (V, Θ) is separable over K if and only if $(V, \Theta)_{\bar{K}}$ is isomorphic to the Fermat form $(\bar{K}^n, \langle 1, \ldots, 1\rangle_r)$ of the same dimension over $\bar{K}$.

(vi) Let L/K be a finite field extension, let (U, Φ) be an r-form over L and let $t \in \mathrm{Hom}_K(L, K)$ such that $t(U, \Phi)$ is separable over K. Then (U, Φ) is separable over L.

Proof: (i) This is clear from Lemma 3.6(ii).
(ii) The first part follows from Lemma 3.6(iii). Let L and M be finite separable field extensions of K, let $b \in L^*, c \in M^*$ and let $g : (L, \mathrm{tr}_{L/K}\langle b\rangle_r) \to (M, \mathrm{tr}_{M/K}\langle c\rangle_r)$ be an isomorphism of r-forms over K. Then

$$\begin{aligned} &\mathrm{tr}_{M/K}(cg(l_1) \cdots g(l_r)) = \mathrm{tr}_{L/K}(bl_1 \cdots l_r) \\ &\quad = \mathrm{tr}_{L/K}(b \cdot 1 \cdot l_1 l_2 \cdot l_3 \cdots l_r) = \mathrm{tr}_{M/K}(cg(1)g(l_1 l_2)g(l_3) \cdots g(l_r)) \end{aligned}$$

for $l_1, \dots, l_r \in L$ and therefore $g(1)g(l_1l_2) = g(l_1)g(l_2)$ by Lemma 3.3(iv). We define $e := g(1)$ and $\phi : L \to M, l \to e^{-1}g(l)$. Then ϕ is a morphism of rings and hence an isomorphism of fields, since $\dim_K(L) = \dim_K(M)$. Then we have

$$\begin{aligned} &\mathrm{tr}_{\mathrm{M/K}}(ce^r\phi(l_1)\cdots\phi(l_r)) = \mathrm{tr}_{\mathrm{M/K}}(cg(l_1)\cdots g(l_r)) \\ &\quad = \mathrm{tr}_{\mathrm{L/K}}(bl_1\cdots l_r) = \mathrm{tr}_{\mathrm{M/K}}(\phi(b)\phi(l_1)\cdots\phi(l_r)), \end{aligned}$$

and therefore $\phi(b) = ce^r$. The reverse implication is clear.
(iii) A sum of separable r-forms is obviously separable. Let (V, Θ) be a separable r-form and let $(V, \Theta) = \sum_i (V_i, \Theta_i)$ be its decomposition into indecomposable r-forms. Then $\mathrm{Cent}(\Theta) = \bigoplus_i \mathrm{Cent}(\Theta_i)$ and each $\mathrm{Cent}(\Theta_i)$ is a separable field extension of K by Lemmas 3.2(ii) and 3.3(iii). By hypothesis, we have $\sum_i \dim_K(V_i) = \dim_K(V) \leq \dim_K(\mathrm{Cent}(\Theta)) = \sum_i \dim_K(\mathrm{Cent}(\Theta_i))$, and by Lemma 3.6(ii), we have $\dim_K(V_i) \geq \dim_K(\mathrm{Cent}(\Theta_i))$ for each i. Hence $\dim_K(V_i) = \dim_K(\mathrm{Cent}(\Theta_i))$, and (V_i, Θ_i) is separable over K for each i.
(iv) By Lemma 3.3(i), the algebra $\mathrm{Cent}(\Theta) \otimes \mathrm{Cent}(\Psi)$ is separable over K if and only if $\mathrm{Cent}(\Theta)$ and $\mathrm{Cent}(\Psi)$ are separable over K. Hence the tensor product of separable r-forms is obviously separable. Now let $(V, \Theta) \otimes (W, \Psi)$ be separable over K. Then we have $\dim_K(\mathrm{Cent}(V)) \cdot \dim_K(\mathrm{Cent}(\Psi)) = \dim_K(V) \cdot \dim_K(W)$. Since $\mathrm{Cent}(\Theta) \otimes \mathrm{Cent}(\Psi) = \mathrm{Cent}(\Theta \otimes \Psi)$ is separable over K, we know that $\mathrm{Cent}(\Theta)$ and $\mathrm{Cent}(\Psi)$ are both separable over K and therefore $\dim_K(\mathrm{Cent}(\Theta)) \leq \dim_K(V)$, $\dim_K(\mathrm{Cent}(\Psi)) \leq \dim_K(W)$ by Lemma 3.6(ii). Hence we have equality of dimensions, hence (V, Θ) and (W, Ψ) are separable over K. (v) follows from Lemmas 3.2(iv) and 3.5(ii). Remark: The scalar extension of an indecomposable separable r-form was explicitly computed in Lemma 1.10(iii).
(vi) follows from Lemmas 3.2(v) and 3.5(iii). □

From the lemma, we get

3.9 Theorem. (The Ring of Separable r-Forms)
The set of separable r-forms is a subring $\hat{W}_r^{sep}(K) \subset \hat{W}_r(K)$. $\hat{W}_r^{sep}(K)$ is the union of the images of the trace maps $tr_{L/K} : \hat{W}_r^D(L) \to \hat{W}_r(K)$, where L/K runs over all finite separable field extensions of K.

4 Cohomological Classification of Separable r-Forms

In the case of quadratic forms, there is an interpretation of the determinant in terms of Galois cohomology, which is given as follows:

Let (V,b) be a quadratic form of dimension n over K. Since every quadratic form is diagonal, there is an isomorphism $(V,b)_{\bar{K}} \cong (\bar{K}^n, \langle 1,\ldots,1\rangle_2)$ of quadratic forms over the separable closure $\bar{K}$. The automorphism group of the quadratic form $(\bar{K}^n, \langle 1,\ldots,1\rangle_2)$ is the orthogonal group O_n. By Weil descent, this gives a bijection between the set of isomorphism classes of quadratic forms of dimension n over K and the cohomology set $H^1(K,\mathrm{O}_n)$. Note that O_n is not an abelian group, so that we are dealing with non-abelian Galois cohomology, and H^1 is not a group, but a pointed set. For a detailed exposition on non-abelian group cohomology, the reader may refer to ([29], Chap. I, §5).

For $n \in \mathbb{N}$, the determinant morphism $\det : \mathrm{O}_n \to \mu_2$ induces a cohomology map $\det : H^1(K,\mathrm{O}_n) \to H^1(K,\mu_2)$. Identifying quadratic forms with the elements of cohomology sets $H^1(K,\mu_2)$ via Weil descent and $H^1(K,\mu_2)$ with K^*/K^{*2} via the Kummer isomorphism, this map coincides with the determinant $\det : \hat{W}(K) \to K^*/K^{*2}$ (cf. [21], §2.4).

We want to apply this technique to r-forms, so we shall recall how the bijections obtained by Weil descent are explicitly given. The following Lemma is cited from ([30], Chap. X, Prop. 2.4):

4.1 Lemma. (Weil Descent)
Let L/K be a Galois field extension and let $\Psi \in \hat{W}_r(L)$ be an r-form over L. Let $E(L/K,\Psi)$ be the set of r-forms over K which become isomorphic to Ψ over L. For $\Theta \in E(L/K,\Psi)$, let $f : \Theta_L \xrightarrow{\sim} \Psi$ be an isomorphism over L. Then the map $a = a(\Theta) : G(L/K) \to \mathrm{Aut}_L(\Psi)$ given by $s \mapsto a_s := f\,{}^s f^{-1}$ is a 1-cocycle and the assignment $\Theta \mapsto a$ induces a bijection

$$E(L/K,\Psi) \leftrightarrow H^1(L/K,\mathrm{Aut}_L(\Psi)).$$

We introduce some notation: Let $n \in \mathbb{N}$, and let A be a set. We write the elements of the n-fold direct sum $A^{\oplus n}$ as vectors $\Sigma_i a_i e_i$ with $a_i \in A$. The left action of the n-th symmetric group S_n is denoted by ${}^\sigma(\Sigma_i a_i e_i) := \Sigma_i a_i e_{\sigma i}$ for $\sigma \in S_n$ and $\Sigma_i a_i e_i \in A^{\oplus n}$.

4.2 Definition. *Let A be a group. The wreath product $S_n \int A$ of S_n with A is defined as the set $S_n \times A^{\oplus n}$ with the semidirect product, which is defined as $(\sigma,a)\cdot(\tau,b) = (\sigma\tau, a\,{}^\sigma b)$ for $\sigma,\tau \in S_n$ and $a,b \in A^{\oplus n}$.*

We have a short exact sequence of groups

$$1 \longrightarrow A^{\oplus n} \longrightarrow S_n \int A \overset{\curvearrowleft}{\longrightarrow} S_n \longrightarrow 1 \tag{1}$$

with a natural splitting given by $\sigma \mapsto (\sigma, 1)$. In the case that $A \subset R^*$ is a subgroup of the multiplicative group of some ring R, we identify $S_n \wr A$ with a subgroup of $GL_n(R)$ by the embedding

$$\begin{array}{rcl} S_n \wr A & \hookrightarrow & GL_n(R), \\ (\sigma, \Sigma_i a_i e_i) & \mapsto & (\delta_{i,\sigma j}\; a_i)_{i,j}. \end{array}$$

where $\delta_{i,j}$ is the Kronecker symbol.

Now let G be a group and let A be a multiplicative G-group. The following notations are taken from Serre's exposition on non-abelian group cohomology in ([29], Chap. I, §5): Consider S_n as a G-group with trivial G-action, and let $\rho \in \mathrm{Hom}(G, S_n)$ be a 1-cocycle. Then ρ induces a twisted G-action on $A^{\oplus n}$ by ${}^{s'}(\Sigma_i a_i e_i) = \Sigma_i a_i e_{\rho_s i}$ (cf. [29], Chap. I, §5.3). This G-module is denoted by ${}_\rho(A^{\oplus n})$. We get a short exact sequence of twisted G-groups

$$1 \longrightarrow {}_\rho(A^{\oplus n}) \longrightarrow {}_\rho(S_n \wr A) \overset{\curvearrowleft}{\longrightarrow} {}_\rho(S_n) \longrightarrow 1,$$

where ${}_\rho(S_n)$ is the set S_n with G-action given by ${}^{s'}\sigma = \rho_s \sigma \rho_s^{-1}$ for $s \in G$, $\sigma \in S_n$. The cocycle $\rho \in \mathrm{Hom}(G, S_n)$ describes a G-action on the set $\{1, \dots, n\}$. Let $n = \Sigma_k n_k$ be the orbit decomposition under this action. This induces decompositions $\rho = \bigoplus_k \rho_k$ and ${}_\rho(A^{\oplus n}) = \bigoplus_k {}_{\rho_k}(A^{\oplus n_k})$ with $\rho_k \in \mathrm{Hom}(G, S_{n_k})$. We say that ρ is transitive if it describes a transitive G-action on the set of n elements, i.e. if $k = 1$ in the above notation.

The following lemma gives a classification of separable K-algebras by Galois cohomology with values in the symmetric group:

4.3 Lemma. *Let $G := G(\bar{K}/K)$ be the absolute Galois group of the field K, and let S_n be the trivial G-module. Then there is a bijection between the cohomology set $H^1(K, S_n)$ and the set of equivalence classes of separable K-algebras of dimension n modulo K-algebra-isomorphism, given as follows: For a transitive cocycle $\rho \in \mathrm{Hom}(G, S_n)$, let $L(\rho) \subset \bar{K}$ be the field fixed by the stabilisator of some point in $\{1, \dots, n\}$. For arbitrary ρ with orbit decomposition $\rho = \bigoplus_k \rho_k$ let $L(\rho) = \bigoplus_k L(\rho_k)$. The inverse map is given by assigning a separable field extension L/K of dimension n to the cocycle corresponding to the action of G_K on the set G_K/G_L.*

Proof: This can be seen as an application of Weil Descent in a more general setting that the one given in Lemma 4.1. In this situation, however, one may simply check that the given maps are inverse bijections. □

4.4 Lemma. *Let V be an indecomposable r-form over K. Then $V_{\bar{K}} \cong W \oplus \cdots \oplus W$ with an indecomposable r-form W over $\bar{K}$. V is separable if and only if W is 1-dimensional over $\bar{K}$.*

Proof: Let $V_{\bar{K}} \cong \bigoplus_{j=1}^{s} \tilde{V}_j^{\oplus m_j}$ be the decomposition into pairwise inequivalent indecomposable r-forms $\tilde{V}_j$ over $\bar{K}$. By Lemma 4.1, $\mathrm{Aut}(V_{\bar{K}}) = \bigoplus_j \mathrm{Aut}(\tilde{V}^{\oplus m_j})$ induces a decomposition $V = \bigoplus_{j=1}^{s} V_j$ of r-forms over K . But V is indecomposable, hence $s = 1$.
Remark: In the special case that (V, Θ) is separable, this was already proved in Lemma 3.8(ii). $\square$

4.5 Lemma. *Let $r \geq 3$ and let (V, Θ) be an indecomposable r-space over K. Then*

$$\mathrm{Aut}\,(V^{\oplus n}, \Theta^{\oplus n}) = S_n \int \mathrm{Aut}\,(V, \Theta) \quad .$$

In particular, the automorphism group of a diagonal r-form of dimension n over $\bar{K}$ is the wreath product $S_n \int \mu_r$.

Proof: Let $A \in \mathrm{Aut}(V^{\oplus n}, \Theta^{\oplus n})$. Write $V^{\oplus n} = Ve_1 \oplus \cdots \oplus Ve_n$ and assume that $A = \sum_{i,j} A_{i,j}$ with $A_{i,j} \in \mathrm{Hom}(Ve_i, Ve_j) \cong \mathrm{End}(V)$. We will show that for every $j \in \{1, \ldots, n\}$ there is $k = k(j) \in \{1, \ldots, n\}$ such that $A_{i,j} = 0$ for $i \neq k$. Since A is an automorphism, this implies that $j \mapsto k(j)$ is a permutation and that $A_{k(j),j} \in \mathrm{Aut}(V)$ for $j = 1, \ldots, n$. Then $A = \sum_i \delta_{i,k(i)} A_{i,k^{-1}(i)} \in S_n \int \mathrm{Aut}(V)$. Let $j \in \{1, \ldots, n\}$ and choose $k \in \{1, \ldots, n\}$ with $A_{k,j} \neq 0$. For $i \neq k$ we have

$$\begin{aligned}
&\Theta(A_{k,j}V, A_{i,j}V, V, \ldots, V) \\
&\quad = \Theta^{\oplus n}(A(Ve_k), A(Ve_i), Ve_j, \ldots, Ve_j) \\
&\quad \subseteq \Theta^{\oplus n}(A(Ve_k), A(Ve_i), A(V^{\oplus n}), \ldots, A(V^{\oplus n})) \\
&\quad = \Theta^{\oplus n}(Ve_k, Ve_i, V^{\oplus n}, \ldots, V^{\oplus n}) = 0.
\end{aligned}$$

If A_{kj} is invertible, then $A_{ij} = 0$ since Θ is regular. Assume A_{kj} is not invertible and let $V' := \mathrm{im}(A_{kj})$, $V'' = \bigoplus_{i \neq k} \mathrm{im}(A_{ij})$. Then the restriction of A gives an isomorphism of r-forms

$$(V, \Theta) \cong (Ve_j, \Theta^{\oplus n}|_{Ve_j}) \xrightarrow{\sim} (V', \Theta^{\oplus n}|_{V'}) \oplus (V'', \Theta^{\oplus n}|_{V''}) \quad .$$

This contradicts the assumption that (V, Θ) is indecomposable. $\square$

For $A \in \mathrm{GL}_n(K)$, $B \in \mathrm{GL}_n(K)$, let

$$A \oplus B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathrm{GL}_{n+m}(K), A \otimes B := \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{pmatrix} \in \mathrm{GL}_{nm}(K).$$

Together with the inclusions $S_n \int \mu_r \subset \mathrm{GL}_n(\bar{K})$, this addition and multiplication induce the structure of a semiring on the set $\bigcup_{n \in \mathbb{N}} H^1(K, S_n \int \mu_r)$. For this we have

4.6 Theorem. *Descent gives an isomorphism of semirings*

$$\hat{W}_r^{+sep}(K) \xrightarrow{\sim} \bigcup_{n \in \mathbb{N}} H^1(K, S_n \int \mu_r).$$

Proof: Immediate from the previous lemmas. $\square$

4.7 Lemma. *Let $(L, tr_{L/K}\langle b\rangle_r)$ be an indecomposable separable r-form of degree n over K. Let $t_1, \ldots, t_n \in G_K$ be a set of representatives for G_K/G_L and choose $\beta \in \bar{K}$ with $\beta^r = b$. Let $\rho \in \mathrm{Hom}(G_K, S_n)$ such that $st_i \in t_{\rho_s i} G_L$ for $s \in G_K$ and $i = 1, \ldots, n$, or equivalently, such that ρ corresponds to L/K under the bijection in Lemma 4.3. Then the image of $(L, tr_{L/K}\langle b\rangle)$ under the bijection in Lemma 4.6 is given by the cocycle*

$$G_K \to S_n \wr \mu_r \quad , \quad s \mapsto \left(\rho_s\, , \sum_i \frac{t_i(\beta)}{st_{\rho_s^{-1} i}(\beta)}\, e_i\right).$$

Proof: Let $l_1, \ldots, l_n$ be a basis of L/K. By Lemma 1.10(ii), there is an isomorphism of r-forms over $\bar{K}$

$$\begin{aligned}(L, \mathrm{tr}_{L/K}\langle b\rangle_r)_{\bar{K}} &\xrightarrow{\sim} (\bar{K}^n, \langle 1, \ldots, 1\rangle_r),\\ l_j \otimes 1 &\mapsto \sum_i t_i(\beta l_j) e_i.\end{aligned}$$

Let $B := (t_i(\beta l_j))$ be its coordinate matrix in the $\bar{K}$-basis $l_1 \otimes 1, \ldots, l_n \otimes 1$ of $L \otimes_K \bar{K}$ and the standard basis of $\bar{K}^n$. By Lemma 4.1, we need to compute the cocycle $s \mapsto B\, {}^sB^{-1}$:

$$\begin{aligned}(B\, {}^sB^{-1})_{ij} &= \sum_{\nu=1}^n B_{i,\nu}\, ({}^sB^{-1})_{\nu,j} = \sum_{\nu=1}^n \tfrac{B_{i,\nu}}{{}^sB_{\rho_s^{-1}i,\nu}}\, {}^sB_{\rho_s^{-1}i,\nu}\, ({}^sB^{-1})_{\nu,j}\\ &= \sum_{\nu=1}^n \tfrac{t_i(\beta l_\nu)}{st_{\rho_s^{-1}i}(\beta l_\nu)}\, {}^sB_{\rho_s^{-1}i,\nu}\, ({}^sB^{-1})_{\nu,j} = \tfrac{t_i(\beta)}{st_{\rho_s^{-1}i}(\beta)}\, s(\sum_{\nu=1}^n B_{\rho_s^{-1}i,\nu}(B^{-1})_{\nu,j})\\ &= \tfrac{t_i(\beta)}{st_{\rho_s^{-1}i}(\beta)}\, \delta_{i,\rho_s j} \quad .\end{aligned}$$

□

4.8 Theorem. *Via the identifications in Lemma 4.3 and Theorem 4.6, the surjective map*

$$\mathrm{Cent} : \bigcup_{n\in\mathbb{N}} H^1(K, S_n \wr \mu_r) \to \bigcup_{n\in\mathbb{N}} H^1(K, S_n)$$

induced by the projections $S_n \wr \mu_r \to S_n$ is the map associating to an r-form its center.

Proof: The projection splits, hence the center map is surjective. From Lemma 3.8, we know that every indecomposable r-form over K with center L is isomorphic to $(L, \mathrm{tr}_{L/K}\langle b\rangle_r)$ for some $b \in L^*$. Lemma 4.7 shows that the cocycle given by projection to S_n is just the one describing L. □

In order to classify separable r-forms using Galois cohomology, we need to compute Galois cohomology of ${}_\rho(S_n)$ and ${}_\rho(\mu_r^{\oplus n})$, where Galois action is twisted by a cocycle $\rho \in \mathrm{Hom}(G_K, S_n)$. On the wreath product $S_n \wr \mu_r$ this action is given as ${}^{s'}(\sigma, \sum_i a_i e_i) = (\rho_s \sigma \rho_s^{-1}, \sum_i {}^s a_i e_{\rho_s i})$ for $s \in G_K$, $\sigma \in S_n$, $a_i \in \mu_r$ and actions on S_n and $\mu_r^{\oplus n}$ are given by projection from this.

4.9 Lemma. (Hilbert 90)
Let M/K be a Galois extension, and let $\rho \in \mathrm{Hom}(G_K, S_n)$. Then we have $H^1(G_{M/K}, {}_\rho(M^{*\oplus n})) = 1$.

Proof: Using direct sum decomposition, we may assume that ρ is transitive. First, let M/K be finite. Write $G = G_{M/K}$. As in the classical proof (cf. [30], Chap. X, §1, Prop. 2) it suffices to show that, given a cocycle $s \mapsto a_s$, there exists $c \in M^{*\oplus n}$ such that the Poincaré sum $b := \sum_{s\in G} a_s {}^{s'}c \in M^{*\oplus n}$ is invertible. Let $a_s = \sum_i (a_s)_i e_i$, $c = \sum_i c_i e_i$, $b = \sum_i b_i e_i$ with $(a_s)_i, c_i, b_i \in M^*$. We have to find c such that $b_i = \sum_s (a_s)_i {}^s c_{\rho_s^{-1} i} \neq 0$ for $i = 1, \ldots, n$. By linear independence of characters, we may choose $c_1 \in M^*$ such that $b_1 = \sum_s (a_s)_1 {}^s c_1 \neq 0$ and let $c_1 = \cdots = c_n := c$. We will show that this implies $b_i \neq 0$ for $i = 2, \ldots, n$. Since ρ is transitive, there exists $t \in G_K$ such that $i = \rho_t 1$. The cocycle condition for a_s with respect to the twisted G_K-action gives $(a_{ts})_i = (a_t)_i {}^t(a_s)_{\rho_t^{-1} i} = (a_t)_i {}^t(a_s)_1$ for $s \in G$. Hence $b_i = \sum_s (a_s)_i {}^s c_1 = \sum_s (a_{ts})_i {}^{ts} c_1 = \sum_s (a_t)_i {}^t(a_s)_1 {}^{ts} c_1 = (a_t)_i {}^t\, b_1 \neq 0$. For an infinite extension, take the direct limit over its finite subextensions. □

4.10 Lemma. *Let $\rho \in \mathrm{Hom}(G_K, S_n)$ be transitive and let L/K be the corresponding field extension. Then*

(i) $H^0(G_K, {}_\rho(\bar{K}^{*\oplus n})) = L^*$.

(ii) $H^1(G_K, {}_\rho(\mu_r^{\oplus n})) = L^*/L^{*r}$.

(iii) $H^0(G_K, {}_\rho(S_n)) = \mathrm{Aut}_K(L)^{\mathrm{opp}}$*, the opposite group.*

Proof: Let $t_1, \ldots, t_n \in G_K$ be representatives for G_K/G_L. For $s \in G_K$ let $\bar{s} \in \mathrm{Hom}(L, \bar{K})$ be the restriction to L. The groups G_K and S_n both act on $\mathrm{Hom}(L, \bar{K}) = \{\bar{t}_1, \ldots, \bar{t}_n\}$ and ρ is given by $\overline{st_i} = \bar{t}_{\rho_s i}$. Assume furthermore that $t_1 = 1 \in G_K$.
(i) Clearly, the map $L^* \to H^0(G_K, {}_\rho(\bar{K}^{*\oplus n}))$ given by $l \mapsto \sum_i t_i(l) e_i$ is injective. Let $a = \sum_i a_i e_i \in H^0(G_K, {}_\rho(\bar{K}^{*\oplus n}))$. Then ${}^{s'}a = a$ for all $s \in G_L$, hence $a_1 \in L^*$, For $i = 1, \ldots, n$ we have ${}^{(t_i)'}a = a$, hence $a_i = t_i(a_1)$. Thus, $a = \sum_i t_i(a) e_i$, hence our map is surjective.
(ii) Like in the classical proof for $H^1(K, \mu_r) = K^*/K^{*r}$, we write down the cohomology sequence for $0 \to {}_\rho(\mu_r^{\oplus n}) \to {}_\rho(\bar{K}^{*\oplus n}) \to {}_\rho(\bar{K}^{*\oplus n}) \to 0$ and use (i) and Hilbert 90.
(iii) For $s \in G_K$ and $\sigma \in S_n$ we have ${}^{s'}\sigma = \rho_s \sigma \rho_s^{-1}$, so the fixed group consists of those permutations which commute with the action of G_K on $\mathrm{Hom}_K(L, \bar{K})$. The action of $\mathrm{Aut}_K(L)$ on $\mathrm{Hom}_K(L, \bar{K})$ by right translation gives an embedding $\mathrm{Aut}_K(L)^{\mathrm{opp}} \hookrightarrow H^0(G_K, {}_\rho(S_n))$. In order to prove surjectivity of this map, let $\sigma \in H^0(G_K, {}_\rho(S_n))$. Then ${}^s({}^\sigma\, \bar{t}_1) = {}^\sigma({}^s\, \bar{t}_1) = {}^\sigma\, \bar{t}_1$ for all $s \in G_L$, which shows that ${}^\sigma\, \bar{t}_1 = \bar{t}_{\sigma 1} \in \mathrm{Aut}_K(L)$. Furthermore, we have ${}^\sigma\, \bar{t}_i = {}^\sigma({}^{t_i}\, \bar{t}_1) = {}^{t_i}({}^\sigma\, \bar{t}_1) = \overline{t_i t_{\sigma 1}}$ for all $i = 1, \ldots, n$, which shows that σ is given by right translation with $t_{\sigma 1}$. □

4.11 Lemma. *Let L/K be a finite separable field extension. Then the isomorphism classes of r-forms with center isomorphic to L correspond bijectively to the orbits in L^*/L^{*r} under the action of* $\mathrm{Aut}_K(L)$.

Proof: This follows from Theorem 4.8, Lemma 4.10, and ([29], Chap. 1, §5.5, Cor. 2 to Prop. 39). □

Remark. We have given a new proof for the classification of separable r-forms in Lemma 3.8(ii).

Now we want to use the cohomological classification for separable r-forms to define first degree cohomological invariants. For this purpose, we study group homomorphism on the wreath product:

4.12 Definition. *Let $M_n(K)$ denote the space of $n \times n$-matrices over the field K. The permanent* $\mathrm{per} : M_n(K) \to K$ *is defined by the modified Leibniz Formula*

$$\mathrm{per}(a_{ij}) := \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma j}.$$

In particular, for $a = (\sigma, (\alpha_1, \ldots, \alpha_n)) \in S_n \int \mu_r$, we have $\mathrm{per}(a) = \prod_{i=1}^{n} \alpha_i$.

4.13 Lemma. *Let $r, n > 0$. Then*

$$\mathrm{Hom}(S_n \int \mu_r, \bar{K}^*) = \mathbb{Z}/(r) \times \mathbb{Z}/(2),$$

where $\mathbb{Z}/(r)$ is generated by the permanent $\mathrm{per} : S_n \int \mu_r \to \mu_r$, *and $\mathbb{Z}/(2)$ is generated by the sign* $\mathrm{sgn} : S_n \int \mu_r \to S_n \xrightarrow{\mathrm{sgn}} \mu_2$.

Proof: From the short exact sequence (1), we obtain the exact sequence of first terms

$$0 \longrightarrow \mathrm{Hom}(S_n, \bar{K}^*) \hookrightarrow \mathrm{Hom}(S_n \int \mu_r, \bar{K}^*) \longrightarrow \mathrm{Hom}(\mu_r^n, \bar{K}^*)^{S_n}.$$

The group $\mathrm{Hom}(S_n, \bar{K}^*)$ is isomorphic to $\mathbb{Z}/2$, generated by the sign, and the group $\mathrm{Hom}(\mu_r^n, \bar{K}^*)^{S_n}$ is isomorphic to $\mathbb{Z}/r$, generated by the product Π. The right arrow splits by $\Pi \mapsto$ per, hence the sequence is a direct product. □

4.14 Definition. (Cohomological Invariants for Separable r-Forms) *We identify separable r-forms with elements of the cohomology sets $H^1(K, S_n \int \mu_r)$ via the bijection in Theorem 4.6 and $H^1(K, \mu_r) = K^*/K^{*r}$ via the Kummer isomorphism.*

(i) Let $\mathrm{per} : \hat{W}_r^{sep}(K) \to K^*/K^{*r}$ *be the cohomology map*

$$\mathrm{per} : H^1(K, S_n \int \mu_r) \to H^1(K, \mu_r).$$

(ii) Let $\mathrm{sgn} : \hat{W}_r^{sep}(K) \to K^*/K^{*2}$ *be the cohomology map*

$$\mathrm{sgn} : H^1(K, S_n \int \mu_r) \to H^1(K, \mu_2).$$

(ii) Let $\left\{ \begin{array}{c} \det : \hat{W}_r^{sep}(K) \to K^*/K^{*r} \\ \det : \hat{W}_r^{sep}(K) \to K^*/K^{*2r} \end{array} \right\}$ *be the cohomology map*

$$\left\{ \begin{array}{c} \det : H^1(K, S_n \int \mu_r) \to H^1(K, \mu_r) \\ \det : H^1(K, S_n \int \mu_r) \to H^1(K, \mu_{2r}) \end{array} \right\} \text{ if } r \text{ is } \left\{ \begin{array}{c} \text{even} \\ \text{odd} \end{array} \right\}.$$

4.15 Lemma. *Let* φ *be one of the invariants* per, sgn, det. *For separable* r*-forms* Θ *and* Ψ *over* K *we have*

$$\varphi(\Theta \oplus \Psi) = \varphi(\Theta) \cdot \varphi(\Psi),$$
$$\varphi(\Theta \otimes \Psi) = \varphi(\Theta)^{dim \Psi} \cdot \varphi(\Psi)^{dim \Theta}.$$

Proof: The equations hold for the respective maps $\bigcup\limits_{n \geq 0} S_n \int \mu_r \to \bar{K}^*$ on the matrix algebra. □

4.16 Lemma. *For an indecomposable separable* r*-form* $(L, tr_{L/K}\langle b \rangle_r)$ *we have*

(i) $\mathrm{per}(L, tr_{L/K}\langle b \rangle_r) = N_{L/K}(b) \in K^*/K^{*r}$.

(ii) $\mathrm{sgn}(L, tr_{L/K}\langle b \rangle_r) = \det(tr_{L/K}) \in K^*/K^{*2}$, *where* $\det(tr_{L/K})$ *is the determinant of the bilinear trace form* $tr_{L/K} : L \times L \to K$.

(iii) $\det(L, tr_{L/K}\langle b \rangle_r) = \left\{ \begin{array}{c} N_{L/K}(b) \det(tr_{L/K})^{r/2} \in K^*/K^{*2} \\ N_{L/K}(b)^2 \det(tr_{L/K})^r \in K^*/K^{*2r} \end{array} \right\}$ *if* r *is* $\left\{ \begin{array}{c} \text{even} \\ \text{odd} \end{array} \right\}$.

Proof: We use the notation from Lemma 4.7. Under the bijection in Theorem 4.6, the r-form $(L, \mathrm{tr}_{L/K}\langle b \rangle_r)$ corresponds to the class of the cocycle $s \mapsto B\, {}^sB^{-1}$ in $H^1(K, S_n \int \mu_r)$, where $B = (t_i(\beta l_j)) \in \mathrm{GL}(\bar{K})$. Hence $\mathrm{per}(B\, {}^sB^{-1})$ is equal to $(\prod_i t_i\beta)\, {}^s(\prod_i t_i\beta)^{-1}$ by Lemma 4.7. Kummer's isomorphism $K^*/K^{*r} \cong H^1(K, \mu_r)$ maps the class of $a \in K^*$ to the class of the cocycle $s \mapsto \alpha\, {}^s\alpha^{-1}$, where $\alpha \in \bar{K}$ is an r-th root of a Hence $\mathrm{per}(L, \mathrm{tr}_{L/K}\langle b \rangle_r) = (\prod_i t_i\beta)^r = \mathrm{N}_{L/K}(b)$, which proves (i).
(ii) By 4.7, $\mathrm{sgn}(L, \mathrm{tr}_{L/K}\langle b \rangle_r)$ does not depend on b, so that we may assume $b = 1$ and have

$$\mathrm{sgn}(L, \mathrm{tr}_{L/K}\langle b \rangle_r) = \mathrm{sgn}(L, \mathrm{tr}_{L/K}\langle 1 \rangle_r) = \det(L, \mathrm{tr}_{L/K}\langle 1 \rangle_r) = \det(B)^2.$$

Now $B^tB = (\mathrm{tr}_{L/K}(l_il_j))$ represents $\mathrm{tr}_{L/K}$ in the basis $l_1, \ldots, l_n$ of L/K, hence $\det(\mathrm{tr}_{L/K}) = \det(B)^2 = \mathrm{sgn}(L, \mathrm{tr}_{L/K}\langle b \rangle_r)$.
(iii) The determinant is the product of the sign and the permanent. The exponents in the formula results from the fact that the cohomology map $K^*/K^{*n} \to K^*/K^{*mn}$ induced by the inclusion $\mu_n \subset \mu_{mn}$ maps the class of a to the class of a^m. □

Example: Classification of separable elliptic curves

Elliptic curves are a well studied family of 3-forms, and it seems natural to compare our classification result for separable 3-forms to the classical classification of elliptic curves.

We recall some basic definitions for elliptic curves: An elliptic curve over K is a non-singular 3-form of dimension 3 with a fixed K-rational point. An isomorphism of elliptic curves is a projective isomorphism of 3-forms respecting the fixed point. Note that, talking about 3-forms, it is understood that the ground field has characteristic $\neq 2, 3$.

4.17 Lemma. *Every elliptic curve over K is isomorphic to an elliptic curve in Weierstrass form*

$$f = x^3 - y^2z - 27c_4xz^2 - 54c_6z^3$$

with $c_4, c_6 \in K$ and $\Delta(f) := (c_4^3 - c_6^2)/12^3 \neq 0$. For an elliptic curve f in Weierstrass form, let $j(f) := c_4^3/\Delta(f)$, called the j-invariant. Then two elliptic curves are isomorphic over $\bar{K}$ if and only if they have the same j-invariant.

Proof: This is found in ([20], Chap. III.2). □

4.18 Lemma. *An elliptic curve is a separable 3-form if and only if its j-invariant is zero. Explicitly, let $f(x) = x^3 - y^2z - 54c_6z^3$ an elliptic curve with $j(f) = 0$, and let $b := 8c_6 \in K^*$. Then*

$$f(x) = x^3 - y^2z - \tfrac{27b}{4}z^3 \cong \left\{ \begin{array}{ll} \langle 1, b, b\rangle_3 & \textit{if } b \in K^{*2} \\ \langle 1\rangle_3 \oplus (K(\sqrt{b}), \mathrm{tr}_{K(\sqrt{b})/K}\langle b\rangle_3) & \textit{if } b \notin K^{*2} \end{array} \right\}.$$

Proof: From 3.8(v) we know that f is separable if and only if it is isomorphic to the Fermat form $f_1 := x^3 + y^3 + z^3$ over $\bar{K}$, and this is isomorphic to the Weierstrass form $f_2 := x^3 - y^2z - \frac{27}{4}z^3$, where the isomorphism $f_2 \xrightarrow{\sim} f_1$ is given by $\varphi := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & -3/2 \\ 0 & -1/3 & -3/2 \end{pmatrix} \in \mathrm{GL}_3(\bar{K})$. Since $j(f_2) = 0$, every separable elliptic curve has j-invariant zero by Lemma 4.17.

We can now use descent to classify elliptic curves over K with $j = 0$ on the one side and separable 3-forms of dimension 3 on the other, and switch from one to the other with φ: The automorphism group $\mathrm{Aut}_{\bar{K}}^{\mathrm{ell}}(f_2)$ of f_2 as an elliptic curve over $\bar{K}$ is μ_6, the group of 6-th roots of unity, where μ_6 acts on $\bar{K}^3$ via $\mu_6 \to \mathrm{GL}(\bar{K})$, $\alpha \mapsto \psi_\alpha := \mathrm{diag}(1, \alpha^{-1}, \alpha^2)$ (cf. [14], Chap. 3, (4.2)). Hence $H^1(K, \mu_6) \cong K^*/K^{*6}$ classifies elliptic curves over K with $j = 0$ by descent (cf. Lemma 4.1) as follows: Let $f = x^3 - y^2z - 54c_6z^3$ an elliptic curve over K with $j(f) = 0$, let $b := 8c_6 \in K^*$, and let $\beta \in \bar{K}$ be a sixth root of b. Then $\psi_\beta := \mathrm{diag}(1, \beta^{-1}, \beta^2) \in \mathrm{GL}(\bar{K})$ gives an isomorphism $f \xrightarrow{\sim} f_2$ over $\bar{K}$, so that f is identified with the class of the cocycle $s \mapsto {\psi_\beta}^s\psi_\beta^{-1} = \psi_{(\beta^s\beta^{-1})}$. Under the Kummer isomorphism, the corresponding cocycle $s \mapsto \beta^s\beta^{-1}$ in $H^1(K, \mu_6)$ is identified with the class of $\beta^6 = b$ in K^*/K^{*r}. The isomorphism $\varphi : f_2 \xrightarrow{\sim} f_1$ gives an embedding

$$\mu_6 = \mathrm{Aut}^{\mathrm{ell}}_{\bar{K}}(f_2) \hookrightarrow \mathrm{Aut}_{\bar{K}}\langle 1, 1, 1\rangle_3 = S_3 \wr \mu_3,$$

$$\alpha \mapsto \varphi\psi_\alpha\varphi^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha^2(1+\alpha^3)/2 & \alpha^2(1-\alpha^3)/2 \\ 0 & \alpha^2(1-\alpha^3)/2 & \alpha^2(1+\alpha^3)/2 \end{pmatrix} = (\sigma^{l(\alpha^3)}, (1, \alpha^2, \alpha^2)) \;,$$

where $\sigma := (23) \in S_3$ and $l : \mu_2 \to \mathbb{Z}/(2)$, $\alpha \to (1+\alpha^3)/2$ is the canonical isomorphism.

The induced cohomology mapping $H^1(K, \mu_6) \to H^1(K, S_3 \wr \mu_3)$ takes the class of the cocycle $s \mapsto \beta^s\beta^{-1}$ to the class of the cocycle $s \mapsto \sigma^{l(\beta^{3s}\beta^{-3})}, (1, \beta^{2s}\beta^{-2}, \beta^{2s}\beta^{-2})$, and it remains to show that this corresponds to the separable 3-form given in the Lemma: Note that β^2 is a third root (β^3 is a square root of b), so that the class of cocycle $s \mapsto \beta^{2s}\beta^{-2}$ ($s \mapsto \beta^{3s}\beta^{-3}$) in $H^1(K, \mu_3)$ (in $H^1(K, \mu_2)$) corresponds to the class of b in K^*/K^{*3} (in K^*/K^{*2}) under the Kummer isomorphism.

Let $b \in K^{*2}$. Then $s \mapsto \beta^{3\,s}\beta^{-3}$ is trivial, so that our cocycle is diagonal and clearly corresponds to the diagonal 3-form $\langle 1, b, b\rangle_3$. Let $b \notin K^{*2}$. Then $s \mapsto \sigma^{l(\beta^{3\,s}\beta^{-3})}$ corresponds to the quadratic extension $K(\sqrt{b})/K$ under the bijection in Lemma 4.3, and our cocycle corresponds to the 3-form $\langle 1\rangle_3 \oplus (K(\sqrt{b}), \mathrm{tr}_{K(\sqrt{b})/K}\langle b\rangle_3)$ by Lemma 4.7. $\square$

5 Cohomological Invariants of Degree 2

Once more, we will study a technique from the theory of quadratic forms in order to obtain analogous results for forms of higher degree. As described in the beginning of Section 4, Weil descent gives a classification for quadratic forms of dimension n over K by the cohomology set $H^1(K, \mathrm{O}_n)$, and in this sense the determinant of quadratic forms is given by the determinant $\det : \mathrm{O}_n \to \mu_2$. Therefore $H^1(K, \mathrm{SO}_n)$ classifies quadratic forms of dimension n and determinant 1:

$$
\begin{tikzcd}
1 \arrow[r] & H^1(K, \mathrm{SO}_n) \arrow[r] \arrow[d, leftrightarrow] & H^1(K, \mathrm{O}_n) \arrow[r] \arrow[d, leftrightarrow] & H^1(K, \mu_2) \arrow[r] \arrow[d, leftrightarrow] & 1 \\
 & \hat{W}(K)_{\substack{dim\,=\,n \\ det\,=\,1}} & \hat{W}(K)_{dim=n} & K^*/K^{*2} &
\end{tikzcd}
$$

Now the semi-simple extension $1 \to \mu_2 \to \mathrm{Spin}_n \to \mathrm{SO}_n \to 1$ induces a long exact cohomology sequence and this gives a degree 2 invariant $H^1(K, \mathrm{SO}_n) \to H^2(K, \mu_2)$ for quadratic forms which is related to the Witt invariant (cf. [21], §2.4).

We want to generalize this construction to forms of degree $r \geq 3$. The idea is as follows: Having in mind the classification of separable r-form by the cohomology set $H^1(K, S_n \int \mu_r)$ from Section 4, we want to find a suitable 'determinant' map $\det : S_n \int \mu_r \to \bar{K}^*$ and let $\mathrm{SO}_{r,n} \subset S_n \int \mu_r$ denote its kernel. Then we construct a central extension of G_K-modules $1 \to \mu_r \to E_{r,n} \to SO_{r,n} \to 1$ and obtain a cohomology map

$$H^1(K, \mathrm{SO}_{r,n}) \to H^2(K, \mu_r)$$

in the long exact cohomology sequence. This gives us a second degree cohomological invariant for separable forms of dimension n and 'determinant' 1.

In 4.13, we saw that the choices for our 'determinant' are basicly the permanent, the sign, and the determinant. Now let r be odd, and let $n \geq 2$. We define subgroups $\mathrm{SO}^{(i)} \subset S_n \int \mu_r$ $(i = 1, 2, 3)$ by the short exact sequences

$$0 \longrightarrow \mathrm{SO}^{(1)}_{r,n} \longrightarrow S_n \int \mu_r \xrightarrow{per} \mu_r \longrightarrow 0, \tag{1}$$

$$0 \longrightarrow \mathrm{SO}^{(2)}_{r,n} \longrightarrow S_n \int \mu_r \xrightarrow{\det} \mu_{2r} \longrightarrow 0, \tag{2}$$

$$0 \longrightarrow \mathrm{SO}^{(3)}_{r,n} \longrightarrow S_n \int \mu_r \xrightarrow{sgn} \mu_2 \longrightarrow 0. \tag{3}$$

In order to construct central extensions, we may first ignore Galois module structure and search for central group extensions. These are classified by the cohomology set $H^2(SO_{r,n}, \mu_r)$, where $SO_{r,n}$ operates trivially on μ_r.

5.1 Lemma. *Let $r \neq 2$ be a prime number, and let μ_r be the trivial $S_n \int \mu_r$-module. Then*

(i) $H^2(SO^{(1)}_{r,n}, \mu_r) = 0$

(ii) Let $r \neq 3$ and $n \geq 4$. Then $H^2(SO^{(2)}_{r,n}, \mu_r) = 0$.

(iv) Let $r \neq 3$ and $n \geq 4$. Then the inflation map induced by the surjection $\mathrm{SO}^{(3)}_{r,n} \xrightarrow{\det} \mu_r$ induces an isomorphism $H^2(SO^{(3)}_{r,n}, \mu_r) \cong \mu_r$.

Before beginning with the proof of the Lemma, we conclude with

5.2 Theorem. (Canonical Central Extension of $\mathrm{SO}^{(3)}$)
Let $r \neq 2, 3$ be a prime number, and let $n \geq 4$. Then there is a canonical central extension of Galois modules

$$0 \longrightarrow \mu_r \longrightarrow \mathrm{Spin}_{n,r} \longrightarrow \mathrm{SO}^{(3)}_{n,r} \longrightarrow 0.$$

This induces a cohomological invariant $\delta : H^1(K, \mathrm{SO}^{(3)}_{n,r}) \to H^2(K, \mu_r)$ of second degree for r-forms of dimension n and sign 1 in the long exact cohomology sequence. The invariant δ vanishes for forms of determinant 1.

Proof: The central group extension described by the generator of $H^2(\mu_r, \mu_r) \cong \mu_r$ is the group μ_{r^2} of r^2-th roots of unity:

$$0 \longrightarrow \mu_r \longrightarrow \mu_{r^2} \longrightarrow \mu_r \longrightarrow 0.$$

The inflation map induced by the surjection $\det : \mathrm{SO}^{(3)}_{r,n} \to \mu_r$ maps this sequence to the pullback along det, given by the commutative diagram of groups with exact rows and columns

$$\begin{tikzcd}
 & & \mathrm{SO}^{(2)}_{n,r} \arrow[r, equal] \arrow[d] & \mathrm{SO}^{(2)}_{n,r} \arrow[d] & \\
0 \arrow[r] & \mu_r \arrow[r] \arrow[d, equal] & \mathrm{Spin}_{n,r} \arrow[r] \arrow[d] & \mathrm{SO}^{(3)}_{n,r} \arrow[r] \arrow[d, "\det"] & 0 \\
0 \arrow[r] & \mu_r \arrow[r] & \mu_{r^2} \arrow[r] & \mu_r \arrow[r] & 0.
\end{tikzcd}$$

Since $\det : S_n \int \mu_r \to \mu_{2r}$ splits, $H^1(K, \mathrm{SO}^{(2)}) \to H^1(K, S_n \int \mu_r)$ is injective and thus equal to the kernel of the determinant map. In other words, forms of determinant 1 are classified by $H^1(K, \mathrm{SO}^{(2)})$. From the diagram it is clear that the composed map $H^1(K, \mathrm{SO}^{(2)}_{n,r}) \to H^1(K, \mathrm{SO}^{(3)}_{n,r}) \xrightarrow{\delta} H^2(K, \mu_r)$ factors through $H^1(K, \mathrm{Spin}_{n,r})$, hence it must vanish. $\square$

Our interest in degree 2 cohomological invariants is in giving a finer classification of those forms, for which the degree 1 invariants in $H^1(K,\mu_r)$ vanish. In odd degree, this is the permanent. But if the sign and the permanent vanish, then so does the determinant, and hence also δ. Thus, there is no new classification here.

For the proof of Lemma 5.1, we will need some Lemmas. This one was given by Bröcker in [3]:

5.3 Lemma. (Homology of the Symmetric and Alternating Group) (Bröcker)
Let p be a prime and let $q = 0$ or 1. Let $\mu_p^{(q)}$ denote the cyclic group of order p considered as an S_n-module by $\sigma a := a^{sgn(\sigma)}$ for $\sigma \in S_n$, $a \in \mu_p$. For a sequence $(a_1,\dots,a_l)$ of integers let $\dim(a_1,\dots,a_l) := \Sigma_{k=1}^l p^{l-k}a_i$ and $\mathrm{rank}(a_1,\dots,a_l) := p^l$ and let $B_{p,q}$ be the set of sequences $(a_1,\dots,a_l)$ of length $l \geq 0$ such that

- $a_1 \geq \cdots \geq a_l > 0$,
- $\left\{ \begin{array}{l} a_k \equiv 0 \text{ or } 1 \mod 2(p-1) \\ a_k \equiv p-1 \text{ or } p-2 \mod 2(p-1) \end{array} \right\}$ *if* $q + a_1 + \cdots + a_{k-1}$ *is* $\left\{ \begin{array}{c} \text{even} \\ \text{odd} \end{array} \right\}$.

Let $F(p,q)$ denote the free $\mathbb{Z}/p$-algebra generated by $B(p,q)$. For a product $x = a^{(1)}\cdots a^{(m)} \in F(p,q)$ of sequences $a^{(i)} = (a_1^{(i)},\dots,a_{l_i}^{(i)}) \in B(p,q)$ let $\dim(x) := \sum_i \dim(a^{(i)})$ and $\mathrm{rank}(x) := \sum_i \mathrm{rank}(a^{(i)})$. Let $U(p,q)$ be the algebra given by $F(p,q)$ modulo the relations $xy = (-1)^{f(x,y)}yx$, where x and y are products of sequences and $f(x,y) := \dim(x)\dim(y) + q\cdot \mathrm{rank}(x)\mathrm{rank}(x)$. Then $H_i(S_n,\mu_p^{(q)})$ is isomorphic to the μ_p-submodule in $U(p,q)$ generated by elements of dimension i and rank n. If $p \neq 2$, then $H_i(A_n,\mu_p) \cong H_i(S_n,\mu_p^{(0)}) \oplus H_i(S_n,\mu_p^{(1)})$.

Proof: The ring $F(p,q)$ is given in ([3], Def. 4.1). The homology of S_n is given in (loc. cit., Theorem 5.8) and that of A_n in (loc. cit., Theorem 7.1). □

5.4 Corollary. *Let p be a prime number and let $n \geq 2$. Then*

$$H_1(S_n,\mu_p^{(q)}) \cong \begin{cases} \mu_2 & \text{for } p=2, n\geq 2, \\ \mu_3 & \text{for } p=3, q=1, n=3,4, \\ 0 & \text{else.} \end{cases}$$

$$H_2(S_n,\mu_p^{(q)}) \cong \begin{cases} \mu_2 & \text{for } p=2, n=2,3, \\ \mu_2\oplus\mu_2 & \text{for } p=2, n\geq 4, \\ \mu_3 & \text{for } p=3, q=1, n=3,4,6,7, \\ 0 & \text{else.} \end{cases}$$

For $p \neq 2$ we have

$$H_1(A_n,\mu_p) \cong \begin{cases} \mu_3 & \text{for } p=3, n=3,4, \\ 0 & \text{else.} \end{cases}$$

$$H_2(A_n,\mu_p) \cong \begin{cases} \mu_3 & \text{for } p=3, n=3,4,6,7, \\ 0 & \text{else.} \end{cases}$$

Proof: $H_i(S_n, \mu_p^{(q)})$ is generated by products of sequences in $B(p,q)$ of dimension i and rank n. Using the relations $xy = (-1)^{f(x,y)}yx$, we find that the set of products in lexicographic order gives a basis. Such a product is zero if and only if $p \neq 2$ and a sequence $a \in B(p,q)$ with $f(a,a) \equiv 1 \bmod 2$ appears twice as a factor, since in this case we have $a^2 = -a^2$. We will count all such products of sequences: We observe that $p \geq i$ in in the considered cases and therefore all sequences appearing have length ≤ 1, for a sequence of length ≥ 2 has rank $> p$. Note that the empty sequence $()$ is an element in $B(p,q)$. The set of products of sequences of length ≤ 1 of dimension i and rank n is $\{()^{n-p}(1)\}$ for $i = 1$ and $\{()^{n-p}(2), ()^{n-2p}\}$ for $i = 2$. One checks that the sequence $()$ is contained in $B(p,q)$ for all p and q and that the sequences $(1), (2)$ are contained in $B(2,0), B(2,1), B(3,1)$ and are not contained in $B(p,q)$ for other p and q. Then one checks that $f((),()) = q$, $f((1),(1)) = 1 + qp^2$ and therefore $()^2 = 0$ for $p \neq 2, q = 1$ and $(1)^2 = 0$ for $p \neq 2, q = 0$. We get

$$
\begin{array}{lclllll}
H_1(S_n, \mu_2^{(q)}) & \cong & ()^{n-2}(1) & \mathbb{Z}/2 & & & \text{for } n \geq 2, \\
H_1(S_n, \mu_3^{(1)}) & \cong & ()^{n-3}(1) & \mathbb{Z}/3 & & & \text{for } n = 3,4, \\
H_2(S_n, \mu_2^{(q)}) & \cong & ()^{n-2}(2) & \mathbb{Z}/2 & & & \text{for } n = 2,3, \\
H_2(S_n, \mu_2^{(q)}) & \cong & ()^{n-2}(1) & \mathbb{Z}/2 \oplus & ()^{n-4}(1)^2 & \mathbb{Z}/2 & \text{for } n \geq 4, \\
H_2(S_n, \mu_3^{(1)}) & \cong & ()^{n-3}(2) & \mathbb{Z}/3 & & & \text{for } n = 3,4, \\
H_2(S_n, \mu_3^{(1)}) & \cong & ()^{n-6}(1)^2 & \mathbb{Z}/3 & & & \text{for } n = 6,7, \\
H_i(S_n, \mu_p^{(q)}) & \cong & 0 & & & & \text{else } (i = 1,2).
\end{array}
$$

□

5.5 Lemma. *Let $n \geq 2$ and let G be an abelian group. Consider G as an S_n-module with trivial action and $G^{\oplus n}$ as an S_n-module in the obvious way. Then there is an isomorphism of S_n-modules*

$$M_{S_n}^{S_{n-1}}(G) \cong G^{\oplus n},$$

where $M_{S_n}^{S_{n-1}}(G)$ denotes the $(S_{n-1} \subset S_n)$-induced module. For $n \geq 3$, we have an analogous isomorphism of A_n-modules

$$M_{A_n}^{A_{n-1}}(G) \cong G^{\oplus n}.$$

Proof: We recall the definition of the induced module: For an inclusion $A \subset B$ of groups and an A-module G, the $(A \subset B)$-induced module $M_B^A(G)$ is the set

$$M_B^A(G) := \{f : B \to G \mid f(ab) = af(b) \text{ for } a \in A, b \in B\}$$

of A-linear maps $f : B \to G$. $M_B^A(G)$ carries a B-action given by $(cf)(b) = f(bc)$ for $b, c \in B$. Let $S_{n-1} \subset S_n$ be given by the inclusion $\{1, \ldots, n-1\} \subset \{1, \ldots, n\}$. Choose $\sigma_1, \ldots, \sigma_n \in S_n$ such that $\sigma_i(i) = n$. Then $\tau \in S_n$ is contained in the coset $S_{n-1}\sigma_i$ if and only if $\tau(i) = n$. We claim that the map

$$M_{S_n}^{S_{n-1}}(G) \to G^{\oplus n} \quad , \quad f \to f(\sigma_1), \ldots, f(\sigma_n)$$

is an isomorphism of S_n-modules. This map is clearly bijective. To show that it is S_n-equivariant, we need to show that $f(\sigma_i\tau) = f(\sigma_{\tau^{-1}i})$ for $\tau \in S_n$ and $f \in M_{S_n}^{S_{n-1}}(G)$. This is equivalent to saying that $\sigma_i\tau$ is contained in the coset $S_{n-1}\sigma_{\tau^{-1}i}$. This is the case, since $\sigma_i\tau\tau^{-1}i = \sigma(i) = n$. The proof for $A_{n-1} \subset A_n$ is analogous. $\square$

5.6 Lemma. *Let A be an abelian group and let $1 \to G' \to G \overset{\curvearrowleft}{\to} G'' \to 1$ be a split exact sequence of groups acting trivially on A. Then the Hochschild Serre spectral sequence gives a filtration $F^2 \subset F^1 \subset F^0 = H^2(G, A)$ by subgroups such that*

$$\begin{aligned} F^2 &= H^2(G'', A), \\ F^1/F^2 &= H^1(G'', H^1(G', A)), \\ F^0/F^1 &= \ker\left(d_2^{0,2} : H^0(G'', H^2(G', A)) \to H^2(G'', H^1(G', A))\right). \end{aligned}$$

Proof: We consider the Hochschild Serre spectral sequence

$$H^p(G'', H^q(G', A)) = E_2^{p,q} \Rightarrow E^{p+q} = H^{p+q}(G, A).$$

The splitting morphism $G'' \to G$ induces splitting morphisms for the edge morphisms $\inf : H^i(G'', A) \to H^i(G, A)$. This means that the edge morphisms are injective, so that the differentials $d_r^{i-r,r-1} : E_r^{i-r,r-1} \to E_r^{i,0}$ vanish for all r and i. Therefore we can express the quotients in the filtration of the limit term $E^2 = H^2(G, A)$ in terms of level 2:

$$E_\infty^{2,0} = E_2^{2,0}\ ,\ E_\infty^{1,1} = E_2^{1,1}\ ,\ E_\infty^{0,2} = E_3^{0,2} = \ker(d_2^{0,2} : E_2^{0,2} \to E_2^{2,1}).$$

5.7 Lemma. *Let $r \neq 2$ be a prime number . Then*

(i) $H^1(S_n, \mu_r) = H^2(S_n, \mu_r) = 0.$

(ii) $H^1(S_n, \mu_r^n) = 0.$

(iii) $H^2(\mathbb{Z}^n, \mu_r) \cong \bigwedge^2 \mu_r^n.$

(iv) $H^0(S_n, \bigwedge^2 \mu_r^n) = 0.$

(v) $H^1(\mu_{r,\Pi=1}^n, \mu_r) = H^1(\mathbb{Z}_{\Sigma=0}^n, \mu_r) = \mu_r^n/\mu_r.$

(vi) $H^1(S_n, \mu_r^n/\mu_r) = 0.$

(vii) $H^0(S_n, H^2(\mathbb{Z}_{\Sigma=0}^n, \mu_r)) = 0.$

(viii) $H^0(S_n, H^2(\mu_{r,\Pi=1}^n, \mu_r)) = 0.$

Throughout the Lemma, S_n can be exchanged by A_n if we assume $r \neq 2, 3$ and $n \geq 4$.

Proof: We prove all statements for S_n. The proof for A_n is analogous if not mentioned otherwise. We begin with a list of short exact sequences which will be used for following computations: We write the group μ_r additively. Let $\mu^n_{r,\Pi=1}$ be defined by the short exact sequence

$$0 \longrightarrow \mu^n_{r,\Pi=1} \longrightarrow \mu^n_r \underset{\Pi}{\overset{\curvearrowleft}{\longrightarrow}} \mu_r \longrightarrow 0 \tag{4}$$

with the non-canonical splitting given by $\alpha \mapsto (\alpha, 0, \ldots, 0)$. Then we have

$$0 \longrightarrow \mathbb{Z}^n_{\Sigma=0} \xrightarrow{r} \mathbb{Z}^n_{\Sigma=0} \longrightarrow \mu^n_{r,\Pi=1} \longrightarrow 0 \tag{5}$$

$$0 \longrightarrow \mathbb{Z}^n_{\Sigma=0} \longrightarrow \mathbb{Z}^n \underset{\Sigma}{\overset{\curvearrowleft}{\longrightarrow}} \mathbb{Z} \longrightarrow 0 \tag{6}$$

with the non-canonical splitting given as in (4).

(i): By Corollary 5.4, we have $H_1(S_n, \mu_r) = H_2(S_n, \mu_r) = 0$. An application of the Coefficient Theorem of Cohomology (cf. [12], Theorem VI.15.1) shows that H^1 and H^2 vanish as well.
(ii): By Lemma 5.5, we have $M^{S_{n-1}}_{S_n}(\mu_r) \cong \mu^n_r$, hence $H^1(S_n, \mu^n_r) = H^1(S_{n-1}, \mu_r)$ by the Shapiro Lemma. This vanishes by (i).
(iii): Using the Künneth Formula for group homology ([12], Theorem VI.15.2) inductively, one proves $H_2(\mathbb{Z}^n, \mathbb{Z}) \cong \bigwedge^2 H_1(\mathbb{Z}^n, \mathbb{Z}) \cong \bigwedge^2 \mathbb{Z}^n$. Then the Coefficient Theorem of Cohomology yields $H^2(\mathbb{Z}^n, \mu_r) \cong \mathrm{Hom}(\bigwedge^2 \mathbb{Z}^n, \mu_r) \cong \bigwedge^2 \mu^n_r$.
(iv): Let $a = \Sigma_{i<j}\, a_{ij}\, e_i \wedge e_j \in \bigwedge^2 \mu^n_r$. Assume that a is S_n-invariant. For $i < j$ let σ be the transposition $(ij) \in S_n$. Then $\sigma a = a$ implies $a_{ij} = -a_{ij}$, hence $a_{ij} = 0$. Now let a be A_n-invariant, and let $i < j$. Choose j, k such that i, j, k, l are pairwise different (note that $n \geq 4$) and let $\sigma := (ij)(kl) \in A_n$. Then $\sigma a = a$ implies $a_{ij} = -a_{ij}$, hence $a_{ij} = 0$.
(v): Apply the functor $Hom(\_, \mu_r)$ to the short exact sequences (4) and (6).
(vi): In the long exact sequence induced by S_n-action on (4) we have

$$\cdots \longrightarrow H^1(S_n, \mu^n_r) \longrightarrow H^1(S_n, \mu^n_r/\mu_r) \longrightarrow H^2(S_n, \mu_r) \longrightarrow \cdots$$

The outer terms vanish by (i) and (ii), hence the middle term also vanishes.
(vii): Applying Lemma 5.6 to the exact sequence (5), trivially acting on μ_r, yields a filtration $F^1 \subset F^0 = H^2(\mathbb{Z}^n, \mu_r)$, and thus a short exact sequence

$$0 \longrightarrow F^1 \longrightarrow H^2(\mathbb{Z}^n, \mu_r) \longrightarrow F^0/F^1 \longrightarrow 0,$$

such that $F^1 = H^1(\mathbb{Z}, H^1(\mathbb{Z}^n_{\Sigma=0}, \mu_r))$ and $F^0/F^1 = H^2(\mathbb{Z}^n_{\Sigma=0}, \mu_r)$. We have $H^1(\mathbb{Z}, H^1(\mathbb{Z}^n_{\Sigma=0}, \mu_r)) = \mu^n_r/\mu_r$ by (v), and $H^2(\mathbb{Z}^n, \mu_r) = \bigwedge^2 \mu^n_r$ by (iii), hence this reads

$$0 \longrightarrow \mu^n_r/\mu_r \longrightarrow \bigwedge^2 \mu^n_r \longrightarrow H^2(\mathbb{Z}^n_{\Sigma=0}, \mu_r) \longrightarrow 0.$$

Now the long exact sequence induced by S_n-action yields

$$\cdots \longrightarrow H^0(S_n, \bigwedge^2 \mu^n_r) \longrightarrow H^0(S_n, H^2(\mathbb{Z}^n_{\Sigma=0}, \mu_r)) \longrightarrow H^1(S_n, \mu^n_r/\mu_r) \longrightarrow \cdots$$

The outer terms vanish by (iv) and (vi), hence the middle terms also vanishes.
(viii): In the Hochschild Serre spectral sequence induced by trivial action of the short exact sequence (5) on μ_r we have the first term exact sequence

$$0 \to H^1(\mu^n_{r,\Pi=1},\mu_r) \to H^1(\mathbb{Z}^n_{\Sigma=0},\mu_r) \to H^1(\mathbb{Z}^n_{\Sigma=0},\mu_r) \to H^2(\mu^n_{r,\Pi=1},\mu_r) \to H^2(\mathbb{Z}^n_{\Sigma=0},\mu_r).$$

By (v), the second arrow is an isomorphism, and the third term is isomorphic to μ^n_r/μ_r. Hence the fix modules under S_n-action give an exact sequence

$$0 \longrightarrow H^0(S_n,\mu^n_r/\mu_r) \longrightarrow H^0(S_n,H^2(\mu^n_{r,\Pi=1},\mu_r)) \longrightarrow H^0(S_n,H^2(\mathbb{Z}^n_{\Sigma=0},\mu_r)).$$

Clearly, the left term is zero, and so is the right one, by (vii). Hence the middle terms is also zero. $\square$

Proof of Lemma 5.1: From Definition 4.2 of the wreath product, we have the split exact sequence

$$0 \longrightarrow \mu^n_r \longrightarrow S_n \int \mu_r \xrightarrow{\curvearrowleft} S_n, \longrightarrow 0. \tag{7}$$

Pullback of this sequence by the inclusions $\mathrm{SO}^{(i)}_{r,n} \subset S_n \int \mu_r$ gives split exact sequences

$$0 \longrightarrow \mu^n_{r,\Pi=1} \longrightarrow \mathrm{SO}^{(1)}_{r,n} \xrightarrow{\curvearrowleft} S_n \longrightarrow 0, \tag{8}$$

$$0 \longrightarrow \mu^n_{r,\Pi=1} \longrightarrow \mathrm{SO}^{(2)}_{r,n} \xrightarrow{\curvearrowleft} A_n \longrightarrow 0, \tag{9}$$

$$0 \longrightarrow \mu^n_r \longrightarrow \mathrm{SO}^{(3)}_{r,n} \xrightarrow{\curvearrowleft} A_n \longrightarrow 0. \tag{10}$$

(i) Apply Lemma 5.6 to the sequence (8) and get a filtration $F^2 \subset F^1 \subset F^0 = H^2(SO^{(1)}_{r,n},\mu_r)$ such that

$$\begin{aligned}
F^2 &= H^2(S_n,\mu_r) \overset{5.7\,(i)}{=} 0,\\
F^1/F^2 &= H^1(S_n,H^1(\mu^n_{r,\Pi=1},\mu_r)) \overset{5.7\,(v)}{=} H^1(S_n,\mu^n_r/\mu_r) \overset{5.7\,(vi)}{=} 0,\\
F^0/F^1 &\subset H^0(S_n,H^2(\mu^n_{r,\Pi=1},\mu_r)) \overset{5.7\,(viii)}{=} 0.
\end{aligned}$$

(ii) is proved with the same argument from the sequence (9).
(iii) Apply Lemma 5.6 to (10) and get a filtration $F^2 \subset F^1 \subset F^0 = H^2(SO^{(3)}_{r,n},\mu_r)$ such that

$$\begin{aligned}
F^2 &= H^2(A_n,\mu_r) \overset{5.7\,(i)}{=} 0,\\
F^1/F^2 &= H^1(A_n,H^1(\mu^n_r,\mu_r)) = H^1(A_n,\mu^n_r) \overset{5.7\,(ii)}{=} 0,\\
F^0/F^1 &\subset H^0(A_n,H^2(\mu^n_r,\mu_r)) = H^0(A_n,H^2(\mu_r,\mu_r)^n) = H^2(\mu_r,\mu_r).
\end{aligned}$$

This gives an injection $H^2(SO^{(3)}_{r,n},\mu_r) \hookrightarrow H^2(\mu_r,\mu_r)$. Now consider the exact sequence

$$0 \longrightarrow \mathrm{SO}^{(2)}_{r,n} \longrightarrow \mathrm{SO}^{(3)}_{r,n} \xrightarrow{\det} \mu_r \longrightarrow 0.$$

From the Hochschild-Serre spectral sequence induced by action on μ_r, we get an exact sequence of first terms

$$H^0(\mu_r, H^1(\mathrm{SO}^{(2)}_{r,n}, \mu_r)) \longrightarrow H^2(\mu_r, \mu_r) \xrightarrow{\text{inf}} H^2(\mathrm{SO}^{(3)}_{r,n}, \mu_r).$$

The group $\mathrm{SO}^{(2)}_{r,n}$ is isomorphic to the semidirect product $A_n \rtimes \mu^n_{r,\Pi=1}$ induced by A_n-action on $\mu^n_{r,\Pi=1}$. Hence we have

$$H^1(\mathrm{SO}^{(2)}_{r,n}, \mu_r) = \mathrm{Hom}(A_n \rtimes \mu^n_{r,\Pi=1}, \mu_r) = \mathrm{Hom}(\mu^n_{r,\Pi=1}, \mu_r)^{A_n} = 0.$$

This shows that the inflation map is injective and thus bijective, since we gave a reverse injection before. $\square$

6 The Generalized Leibniz Formula

In this section, we study a generalization of the Leibniz formula for the quadratic determinant in order to obtain an invariant for higher degree forms.

Let V be a K-vector space of dimension n and let $r \geq 2$ be an integer. An r-multilinear (not necessarily symmetric) form on V is an r-fold tensor $\Theta \in (V^{\otimes r})^* = (V^*)^{\otimes r}$. In the case $r = 2$ the determinant of bilinear forms in the map $\det : V^* \otimes V^* \to K$ which is uniquely defined up to multiple by the equivalent conditions

(i) The determinant is multilinear and alternating in the first argument,

(ii) The determinant is multilinear and alternating in the second argument.

and after choosing a K-basis $\{v_1, \ldots, v_n\}$ of V, it is computed by the Leibniz formula

$$\det(\Theta) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \Theta(v_i, v_{\sigma i}).$$

6.1 Definition.

(i) For $k = 1, \ldots, r$, the action of $\operatorname{End}(V)$ on the k-th argument induces a (right) action on the space of multilinear r-forms on V which we write

$$(\Theta \circ_k \varphi)(v_1, \ldots, v_r) := \Theta(v_1, \ldots, v_{k-1}, \varphi v_k, v_{k+1}, \ldots, v_r)$$

for $\Theta \in (V^{\otimes r})^*$, $\varphi \in \operatorname{End}(V)$. We say that a map $D : (V^{\otimes r})^* \to K$ is multilinear and alternating in direction k if we have $D(\Theta \circ_k \varphi) = \det(\varphi) \cdot D(\Theta)$ for $\Theta \in (V^{\otimes r})^*$, $\varphi \in \operatorname{End}(V)$.

(ii) The actions $\circ_1, \ldots, \circ_r$ from (i) are pairwise commutative. Let

$$(\Theta \circ \varphi)(v_1, \ldots, v_r) := (\Theta \circ_1 \varphi \circ_2 \varphi \cdots \circ_r \varphi)(v_1, \ldots, v_r) = \Theta(\varphi v_1, \ldots, \varphi v_r)$$

denote the simultaneous action on all arguments of Θ. In particular, equivalence of r-form means equivalence under the restriction of this action to $\operatorname{GL}(V)$.

6.2 Lemma. *Let $\{v_1, \ldots, v_n\}$ be a K-basis of V and let $\det' : (V^{\otimes r})^* \to K$ be given by the generalized Leibniz formula*

$$\det'(\Theta) := \sum_{\sigma_2, \ldots, \sigma_r \in S_n} \operatorname{sgn}(\sigma_2 \cdots \sigma_r) \prod_{i=1}^{n} \Theta(v_i, v_{\sigma_2 i}, \ldots, v_{\sigma_r i}).$$

(i) $\det'$ is multilinear and alternating in directions $2, \ldots, r$.

(ii) If r is even, then $\det'$ is invariant under S_r-action. In particular, it follows from (i) that $\det'$ is multilinear and alternating in direction 1.

(iii) Let r be odd, and let $\Theta \in (V^{\otimes r})^*$ be the diagonal form given by

$$\Theta(v_{i_1}, \ldots, v_{i_r}) := \left\{ \begin{array}{ll} 1 & \textit{if } i_1 = \cdots = i_r \\ 0 & \textit{else} \end{array} \right\}.$$

Then $\det'(\Theta \circ \varphi) = \operatorname{per}(\varphi) \cdot \det(\varphi)^{r-1}$, *where* $\operatorname{per}(\varphi)$ *is the permanent from Definition 4.12.*

Proof: (i) Let $\Theta \in (V^{\otimes r})^*$ and let $\varphi \in \operatorname{End}(V)$ be given by $\varphi v_i = \Sigma_j a_{ij} v_j$. Then

$$\begin{aligned}
\det'(\Theta \circ_2 \varphi) &= \sum_{\sigma_2,\ldots,\sigma_r} \operatorname{sgn}(\sigma_2 \cdots \sigma_r) \prod_{i=1}^n \Theta(v_i, \varphi v_{\sigma_2 i}, v_{\sigma_3 i}, \ldots, v_{\sigma_r i}) \\
&= \sum_{\sigma_2,\ldots,\sigma_r} \operatorname{sgn}(\sigma_2 \cdots \sigma_r) \prod_{i=1}^n \left(\sum_{\nu=1}^n a_{\sigma_2 i, \nu} \Theta(v_i, v_\nu, v_{\sigma_3 i}, \ldots, v_{\sigma_r i}) \right) \\
&= \sum_{\sigma_2,\ldots,\sigma_r} \operatorname{sgn}(\sigma_2 \cdots \sigma_r) \sum_{\nu_1=1}^n \cdots \sum_{\nu_n=1}^n \left(\prod_{i=1}^n a_{\sigma_2 i, \nu_i} \Theta(v_i, v_\nu, v_{\sigma_3 i}, \ldots, v_{\sigma_r i}) \right) \\
&= \sum_{\sigma_3,\ldots,\sigma_r} \operatorname{sgn}(\sigma_3 \cdots \sigma_r) \sum_{\nu_1=1}^n \cdots \sum_{\nu_n=1}^n \left(\sum_{\sigma_2 \in S_n} \operatorname{sgn}(\sigma_2) \prod_{i=1}^n a_{\sigma_2 i, \nu_i} \right) \prod_{i=1}^n \Theta(v_i, v_{\nu_i}, v_{\sigma_3 i}, \ldots, v_{\sigma_r i}) \\
&= \sum_{\sigma_3,\ldots,\sigma_r} \operatorname{sgn}(\sigma_3 \cdots \sigma_r) \sum_{\nu_1=1}^n \cdots \sum_{\nu_n=1}^n \det(a_{i,\nu_j}) \prod_{i=1}^n \Theta(v_i, v_{\nu_i}, v_{\sigma_3 i}, \ldots, v_{\sigma_r i}).
\end{aligned}$$

Now $\det(a_{i,\nu_j})$ vanishes unless $(\nu_1, \ldots, \nu_n)$ is a permutation. Hence this is equal to

$$\begin{aligned}
&\sum_{\sigma_3,\ldots,\sigma_r} \operatorname{sgn}(\sigma_3 \cdots \sigma_r) \sum_{\sigma_2 \in S_n} \det(a_{i,\sigma_2 i}) \prod_{i=1}^n \Theta(v_i, v_{\sigma_2 i}, v_{\sigma_3 i}, \ldots, v_{\sigma_r i}) \\
&\quad = \sum_{\sigma_2,\ldots,\sigma_r} \operatorname{sgn}(\sigma_3 \cdots \sigma_r) \operatorname{sgn}(\sigma_2) \det(a_{ij}) \prod_{i=1}^n \Theta(v_i, v_{\sigma_2 i}, v_{\sigma_3 i}, \ldots, v_{\sigma_r i}) \\
&\quad = \det(\varphi) \cdot \det'(\Theta).
\end{aligned}$$

The same computation works for the directions $3, \ldots, r$.

(ii): Let $\tau \in S_r$, let $\sigma_2, \ldots, \sigma_r \in S_n$, and let $\sigma_1 := 1 \in S_n$ denote the identity. Then

$$\prod_{i=1}^n {}^\tau\Theta(v_{\sigma_1 i}, v_{\sigma_2 i}, \ldots, v_{\sigma_r i}) = \prod_{i=1}^n \Theta(v_{\sigma_{\tau 1} i}, v_{\sigma_{\tau 2} i}, \ldots, v_{\sigma_{\tau r} i}) = \prod_{i=1}^n \Theta(v_i, v_{\sigma_{\tau 2}\sigma_{\tau 1}^{-1} i}, \ldots, v_{\sigma_{\tau r}\sigma_{\tau 1}^{-1} i}).$$

Now the map $S_n^{r-1} \to S_n^{r-1}$ given by $(\sigma_2, \ldots, \sigma_r) \mapsto (\sigma_{\tau 2}\sigma_{\tau 1}^{-1}, \ldots, \sigma_{\tau r}\sigma_{\tau 1}^{-1})$ is a permutation, and $\operatorname{sgn}(\sigma_{\tau 2}\sigma_{\tau 1}^{-1} \cdots \sigma_{\tau r}\sigma_{\tau 1}^{-1}) = \operatorname{sgn}(\sigma_{\tau 1} \cdots \sigma_{\tau r}) \cdot \operatorname{sgn}(\sigma_{\tau 1})^{r-2} = \operatorname{sgn}(\sigma_2 \cdots \sigma_r)$, so that we have

$$\begin{aligned}
\det'({}^\tau\Theta) &= \sum_{\sigma_2,\ldots,\sigma_r} \operatorname{sgn}(\sigma_2 \cdots \sigma_r) \prod_{i=1}^n {}^\tau\Theta(v_{\sigma_1 i}, v_{\sigma_2 i}, \ldots, v_{\sigma_r i}) \\
&= \sum_{\sigma_2,\ldots,\sigma_r} \operatorname{sgn}(\sigma_{\tau 2}\sigma_{\tau 1}^{-1} \cdots \sigma_{\tau r}\sigma_{\tau 1}^{-1}) \prod_{i=1}^n \Theta(v_i, v_{\sigma_{\tau 2}\sigma_{\tau 1}^{-1} i}, \ldots, v_{\sigma_{\tau r}\sigma_{\tau 1}^{-1} i}) = \det'(\Theta)\ .
\end{aligned}$$

(iii): $\det'(\Theta \circ \varphi) = \sum_{\sigma_2,\dots,\sigma_r} \operatorname{sgn}(\sigma_2 \cdots \sigma_r) \prod_{i=1}^{n} \Theta(\varphi v_i, \varphi v_{\sigma_2 i}, \dots, \varphi v_{\sigma_r i})$

$$
\begin{aligned}
&= \sum_{\sigma_2,\dots,\sigma_r} \operatorname{sgn}(\sigma_2 \cdots \sigma_r) \prod_{i=1}^{n} \sum_{\nu_1=1}^{n} \cdots \sum_{\nu_r=1}^{n} a_{i,\nu_1} a_{\sigma_2 i,\nu_2} \cdots a_{\sigma_r i,\nu_r}\ \Theta(v_{\nu_1}, \dots, v_{\nu_r}) \\
&= \sum_{\sigma_2,\dots,\sigma_r} \operatorname{sgn}(\sigma_2 \cdots \sigma_r) \prod_{i=1}^{n} \sum_{\nu=1}^{n} a_{i,\nu} a_{\sigma_2 i,\nu} \cdots a_{\sigma_r i,\nu} \\
&= \sum_{\nu_1=1}^{n} \cdots \sum_{\nu_n=1}^{n} \sum_{\sigma_2,\dots,\sigma_r} \operatorname{sgn}(\sigma_2 \cdots \sigma_r) \prod_{i=1}^{n} a_{i,\nu_i} a_{\sigma_2 i,\nu_i} \cdots a_{\sigma_r i,\nu_i} \\
&= \sum_{\nu_1=1}^{n} \cdots \sum_{\nu_n=1}^{n} \prod_{i=1}^{n} a_{i,\nu_i} \prod_{k=2}^{r} \left(\sum_{\sigma_k \in S_n} \operatorname{sgn}(\sigma_k) \prod_{i=1}^{n} a_{\sigma_k i,\nu_i} \right) \\
&= \sum_{\nu_1=1}^{n} \cdots \sum_{\nu_n=1}^{n} \prod_{i=1}^{n} a_{i,\nu_i}\ \det(a_{i,\nu_j})^{r-1}
\end{aligned}
$$

Now $\det(a_{i,\nu_j})$ vanishes unless $(\nu_1, \dots, \nu_n)$ is a permutation, hence this is equal to

$$
\sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma_i}\ \det(a_{i,\sigma j})^{r-1} = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma_i} \left(\operatorname{sgn}(\sigma) \det(\varphi) \right)^{r-1}.
$$

Now $\operatorname{sgn}(\sigma)^{r-1} = 1$, since r is odd, hence this is equal to $\operatorname{per}(\varphi) \det(\varphi)^{r-1}$. □

Remark. A similar result was given by Cayley in ([5], p.86). See also ([27], §54) and [18]. We obtain the following

6.3 Theorem. *The generalized Leibniz formula given in Lemma 6.2 induces an invariant*

$$
\det' : \hat{W}_r^+(K) \to K/K^{*r}
$$

for r-forms if and only if r is even.

Proof: If r is even, then $\det'$ is multilinear and alternating in each of the directions $1, \dots, r$, hence we have $\det'(\Theta \circ \varphi) = \det(\varphi)^r \cdot \det'(\Theta)$ for $\Theta \in (V^{\otimes r})^*$, $\varphi \in \operatorname{End}(V)$. This shows that the class of $\det'(\Theta)$ in K/K^{*r} is independent from the choice of the basis in Lemma 6.2, and that it is well defined for r-forms of even degree.
Let r be odd. By Lemma 6.2(iii), the values of $\det'$ on the equivalence class of the diagonal r-form $\langle 1, \dots, 1 \rangle_r$ are given by the set $\{\operatorname{per}(A) \det(A)^{r-1} \mid A \in \mathrm{GL}_n(K)\}$. But for each $x \in K$ there is an $A \in \mathrm{SL}(K)$ such that $\operatorname{per}(A) = x$, hence the class of $\det'\langle 1, \dots, 1 \rangle_r$ in K/K^{*r} is not well defined. □

We want to compare $\det'$ to the determinant det defined in Lemma 4.14:

6.4 Theorem. *Let $r > 2$ be even, and let Θ be a separable r-form over K. Then*

$$
\det'(\Theta) = \det(\Theta).
$$

In the following sections, we will therefore use the following

6.5 Notation. *Let r be even. The theorem allows us to write*

$$\det := \det' : \hat{W}(K) \to K/K^{*r}.$$

Proof of Theorem 6.4: Let $(L, \mathrm{tr}_{\mathrm{L/K}}\langle b\rangle_r)$ be an indecomposable separable r-form over K. Let $\{l_1, \ldots, l_n\}$ be a K-Basis of L and let $\mathrm{Hom}_K(L, \bar{K}) = \{t_1, \ldots, t_n\}$. Then

$$\begin{aligned} &\det'(L, \mathrm{tr}_{\mathrm{L/K}}\langle b\rangle_r) = \sum_{\sigma_2,\ldots,\sigma_r} \mathrm{sgn}(\sigma_2 \cdots \sigma_r) \prod_{i=1}^{n} \left(\sum_{\nu=1}^{n} t_\nu(b\, l_i\, l_{\sigma_2 i} \cdots l_{\sigma_r i}) \right) \\ &= \sum_{\nu_1=1}^{n} \cdots \sum_{\nu_n=1}^{n} \sum_{\sigma_2,\ldots,\sigma_r} \mathrm{sgn}(\sigma_2 \cdots \sigma_r) \prod_{i=1}^{n} t_{\nu_i}(b\, l_i\, l_{\sigma_2 i} \cdots l_{\sigma_r i}) \end{aligned} \tag{1}$$

Now we show that $\sum\limits_{\sigma_2,\ldots,\sigma_r} \mathrm{sgn}(\sigma_2 \cdots \sigma_r) \prod\limits_{i=1}^{n} t_{\nu_i}(b\, l_i\, l_{\sigma_2 i} \cdots l_{\sigma_r i})$ vanishes unless the vector $(\nu_1, \ldots, \nu_n)$ is a permutation. In this case, we may assume without loss of generality that $\nu_1 = \nu_2$, and let π denote the transposition $(12) \in S_n$. Then $\nu_i = \nu_{\pi i}$ for $i = 1, \ldots, n$ and for $\sigma_2, \ldots, \sigma_r \in S_n$ we have

$$\begin{aligned} &\prod_{i=1}^{n} t_{\nu_i}(b\, l_i\, l_{\sigma_2 i} \cdots l_{\sigma_r i}) = \prod_{i=1}^{n} t_{\nu_i}(b\, l_i) \prod_{k=2}^{r} \prod_{i=1}^{n} t_{\nu_i}(l_{\sigma_k i}) = \prod_{i=1}^{n} t_{\nu_i}(b\, l_i) \prod_{k=2}^{r} \prod_{i=1}^{n} t_{\nu_{\pi i}}(l_{\sigma_k \pi i}) \\ &= \prod_{i=1}^{n} t_{\nu_i}(b\, l_i) \prod_{k=2}^{r} \prod_{i=1}^{n} t_{\nu_i}(l_{\sigma_k \pi i}) = \prod_{i=1}^{n} t_{\nu_i}(b\, l_i\, l_{\sigma_2 \pi i} \cdots l_{\sigma_r \pi i}) \; . \end{aligned}$$

The map $S_n^{r-1} \to S_n^{r-1}$, $(\sigma_2, \ldots, \sigma_r) \mapsto (\sigma_2\pi, \ldots, \sigma_r\pi)$ is a permutation, and we have $\mathrm{sgn}(\pi)^{r-1} = -1$, since r is even. Therefore

$$\begin{aligned} &\sum_{\sigma_2,\ldots,\sigma_r \in S_n} \mathrm{sgn}(\sigma_2 \cdots \sigma_r) \prod_{i=1}^{n} t_{\nu_i}(b\, l_i\, l_{\sigma_2 i} \cdots l_{\sigma_r i}) \\ &= \sum_{\sigma_2,\ldots,\sigma_r \in S_n} \mathrm{sgn}(\sigma_2\pi \cdots \sigma_r\pi) \prod_{i=1}^{n} t_{\nu_i}(b\, l_i\, l_{\sigma_2 \pi i} \cdots l_{\sigma_r \pi i}) \\ &= -\sum_{\sigma_2,\ldots,\sigma_r \in S_n} \mathrm{sgn}(\sigma_2 \cdots \sigma_r) \prod_{i=1}^{n} t_{\nu_i}(b\, l_i\, l_{\sigma_2 i} \cdots l_{\sigma_r i}) = 0. \end{aligned}$$

Hence (1) is equal to

$$\begin{aligned} &\sum_{\sigma \in S_n} \sum_{\sigma_2,\ldots,\sigma_r} \mathrm{sgn}(\sigma_2 \cdots \sigma_r) \prod_{i=1}^{n} t_{\sigma^{-1} i}(b\, l_i\, l_{\sigma_2 i} \cdots l_{\sigma_r i}) \\ &= \sum_{\sigma \in S_n} \sum_{\sigma_2,\ldots,\sigma_r} \mathrm{sgn}(\sigma_2 \cdots \sigma_r) \prod_{i=1}^{n} t_i(b\, l_{\sigma i} l_{\sigma_2 \sigma i} \cdots l_{\sigma_r \sigma i}) \end{aligned}$$

The mapping $S_n^r \to S_n^r$, $(\sigma, \sigma_2, \ldots, \sigma_r) \mapsto (\sigma, \sigma_2\sigma, \ldots, \sigma_r\sigma)$ is a permutation, and we have $\mathrm{sgn}(\sigma \cdot \sigma_2\sigma \cdots \sigma_r\sigma) = \mathrm{sgn}(\sigma_2 \cdots \sigma_r) \cdot \mathrm{sgn}(\sigma)^r = \mathrm{sgn}(\sigma_2 \cdots \sigma_r)$. Hence the sum is equal to

$$\begin{aligned} &N_{\mathrm{L/K}}(b) \sum_{\sigma_1,\ldots,\sigma_r} \mathrm{sgn}(\sigma_1 \cdots \sigma_r) \prod_{i=1}^{n} \prod_{k=1}^{r} t_i(l_{\sigma_k i}) \\ &= N_{\mathrm{L/K}}(b) \cdot \det(t_i(l_\nu))^r = \det(L, \mathrm{tr}_{\mathrm{L/K}}\langle b\rangle_r). \end{aligned}$$

□

7 Discriminants

Let $\mathrm{char}(K) = 0$. The following definition is cited from ([9], Chap. 1, 1.B):

7.1 Definition. *Let V be a finite-dimensional K-vector space and let $\mathbb{P} = \mathbb{P}(V)$ be its projective space. Let $X \subset \mathbb{P}$ be an irreducible closed algebraic variety. Let $X^\vee \subset \mathbb{P}^*$ be the projectively dual variety. Then $X^\vee$ is irreducible. Let Δ_X denote the X-discriminant, which is defined as follows:*

If codim$(X^\vee) = 1$, then Δ_X is the defining polynomial of $X^\vee$.

If codim$(X^\vee) > 1$, then $\Delta_X = 1$.

Note that Δ_X is defined only up to a non-zero constant multiple. There are two interesting ways to obtain invariants for r-forms from this definition. We present them here and compare them to the invariants studied before:

The Discriminant

As defined before, let $I(r,n) \subset \mathbb{N}^n$ be the set of non-negative n-tuples of weight r, and let $\{x^\nu := x_1^{\nu_1} \cdots x_n^{\nu_n} \mid \nu = (\nu_1, \dots, \nu_n) \in I(r,n)\}$ be the set of monomials of degree r in n variables $x_1, \dots, x_n$.

Let $V = K^n$ and let $X = \mathbb{P}(V) \hookrightarrow \mathbb{P}(S^r V)$ be the Veronese embedding. The space $(S^r V)^*$ is identified with the space of r-forms $f = \sum\limits_{\nu \in I(r,n)} a_\nu x^\nu$ of degree r in n variables, and $X^\vee \subset \mathbb{P}(S^r V)^*$ consists of the forms f for which the hypersurface $(f = 0) \subset \mathbb{P}(V)$ is singular. Let $\Delta_{r,n} := \Delta_X$ be the X-discriminant.

7.2 Lemma.

(i) For an r-form $f = \sum\limits_{\nu \in I(r,n)} a_\nu x^\nu$ over K, the discriminant $\Delta_{r,n}(f)$ is a polynomial expression in the coefficients a_ν and homogeneous of degree $n(r-1)^{n-1}$. It is uniquely defined in $\mathbb{Z}_r[a_\nu]$ by the requirement that $\Delta_{r,n}(x_1^r + \cdots + x_n^r) = 1$.

(ii) Let f be an r-form, let $\varphi \in \mathrm{End}(V)$, and let $f \circ \varphi(x) = f(\varphi x)$ (cf. 6.1). Then

$$\Delta_{r,n}(f \circ \varphi) = \Delta_{r,n}(f) \cdot \det(\varphi)^{r(r-1)^{n-1}}.$$

(iii) Each monomial $\prod\limits_{\nu \in I(r,n)} a_\nu^{m_\nu}$ occuring in $\Delta_{r,n}$ satisfies the equations

$$\sum_\nu m_\nu \cdot \nu_i = r(r-1)^{n-1} \quad (i = 1, \dots, n).$$

(iv) Let f and g be r-forms with $\dim(f) = n$, $\dim(g) = m$. Then

$$\Delta_{r,n+m}(f \oplus g) = \Delta_{r,n}(f)^{(r-1)^m} \cdot \Delta_{r,m}(g)^{(r-1)^n}.$$

Proof: (i) is shown in ([9], Chap. 13, 1.D). There, the discriminant is uniquely defined by the requirement that it is irreducible over $\mathbb{Z}$. For this normalization, it is shown in ([9], Chap. 13, Prop. 1.7, (1.10)) that the discriminant of the diagonal r-form $x_1^r + \cdots + x_n^r$ is equal to $\pm r^\alpha$, where

$$\alpha = \alpha(r,n) = \sum_{i=0}^{n-1} (-1)^i \tbinom{n}{i} (n-1-i) r^{n-1-i}.$$

Hence we have coefficients in $\mathbb{Z}_r$ in our normalization.
(ii) The expressions $\Delta_{r,n}(f \circ \varphi)$ and $\Delta_{r,n}(f) \cdot \det(\varphi)$ considered as polynomials in the coefficients of f and φ, have the same zeros, so by the Nullstellensatz $\Delta_{r,n}(f \circ \varphi) = \Delta_{r,n}(f)^p \cdot \det(\varphi)^q$ with $p, q \in \mathbb{N}$. Comparing the homogeneous degree given in (i) on both sides we can determine p and q: For the coefficients $\{a_\nu \,|\, \nu \in I(r,n)\}$ of f we have

$$\begin{aligned}
&p \cdot n(r-1)^{n-1} = p \cdot \deg_{\{a_\nu\}}(\Delta(f)) + q \cdot \deg_{\{a_\nu\}}(\det(\varphi)) \\
&= \deg_{\{a_\nu\}}(\Delta(f)^p \cdot \det(\varphi)^q) = \deg_{\{a_\nu\}}(\Delta(f \circ \varphi)) \\
&= \deg_{\{a_\nu\}}(\Delta(f)) \cdot \deg_{\{a_\nu\}}(f \circ \varphi) = n(r-1)^{n-1},
\end{aligned}$$

and thus $p = 1$. For the coefficients $\{a_{ij}\}$ of φ we have

$$\begin{aligned}
&q \cdot n = p \cdot \deg_{\{a_{ij}\}}(\Delta(f)) + q \cdot \deg_{\{a_{ij}\}}(\det(\varphi)) \\
&= \deg_{\{a_{ij}\}}(\Delta(f)^p \cdot \det(\varphi)^q) = \deg_{\{a_{ij}\}}(\Delta(f \circ \varphi)) \\
&= \deg_{\{a_{ij}\}}(\Delta(f)) \cdot \deg_{\{a_{ij}\}}(f \circ \varphi) = n(r-1)^{n-1} \cdot r,
\end{aligned}$$

and thus $q = r(r-1)^{n-1}$.
(iii) By ([9], Chap.9, Prop. 1.3), the vector $\sum_\nu m_\nu \cdot \nu \in \mathbb{N}^n$ is the same for all monomials $\prod_\nu a_\nu^{m_\nu}$ occuring in $\Delta_{r,n}$. Now let $a_1, \ldots, a_n \in K$ and consider the diagonal r-form $\langle a_1, \ldots, a_n \rangle_r = \sum_i a_i x_1^r$. Over $\bar{K}$, we have $\langle a_1, \ldots, a_n \rangle_r = \langle 1, \ldots, 1 \rangle_r \circ \varphi$, where $\alpha_i \in \bar{K}$ is an r-th root of a_i and $\varphi = \varphi(\alpha_1, \ldots, \alpha_n) \in \mathrm{End}(V)$ is given by $x_i \mapsto \alpha_i x_i$. Now (ii) gives

$$\Delta_{r,n}\langle a_1, \ldots, a_n \rangle_r = \Delta_{r,n}\langle 1, \ldots, 1 \rangle_r \cdot \det(\varphi)^{r(r-1)^{n-1}} = \textstyle\prod_i a_i^{(r-1)^{n-1}}.$$

This shows that the monomial $\prod\limits_{i=1}^{n} m_{re_i}^{(r-1)^{n-1}}$ occurs in $\Delta_{r,n}$, where re_i is the r-fold multiple of the i-th unit vector. Therefore $\sum_\nu m_\nu \cdot \nu_i = r(r-1)^{n-1}$ for $i = 1, \ldots, n$.
(iv) By the Nullstellensatz again, we have $\Delta_{r,n+m}(f \oplus g) = \Delta_{r,n}(f)^p \cdot \Delta_{r,m}(g)^q$ with $p, q \in \mathbb{N}$ and putting in diagonal forms $f = \langle a_1, \ldots, a_n \rangle_r$ and $g = \langle b_1, \ldots, b_m \rangle_r$ we get

$$(\textstyle\prod_i a_i \prod_j b_j)^{(r-1)^{n+m-1}} = \Delta(f \oplus g) = \Delta(f)^p \cdot \Delta(g)^q = \prod_i a_i^{p(r-1)^{n-1}} \prod_j b_j^{q(r-1)^{m-1}}$$

and therefore $p = (r-1)^m$, $q = (r-1)^n$. $\square$

Remark. In [9], all this is done for $K = \mathbb{C}$. It is also true for any field of characteristic 0, since we can restrict to the field $\mathbb{Q}(a_\nu)$ containing the coefficients of f and embed this into $\mathbb{C}$. For a field of characteristic p with $(p, r) = 1$ we believe that the lemma is still true if we replace $\mathbb{Z}$ by the prime field.

7.3 Lemma. *Let (V, Θ) be a multilinear r-form over K with basis $v_1, \ldots, v_n$ and let $f \in K[x_1, \ldots, x_n]$ be the homogeneous r-form given by $\Theta \xleftrightarrow{\{v_i\}} f$ (cf. Lemma 2.1). Then the following statements (i) to (iii) are equivalent:*

(i) The discriminant $\Delta_{r,n}(f)$ is non-zero.

(ii) The homogeneous r-form f is non-singular.

(iii) The multilinear r-form Θ is $(r-1)$-regular over the separable closure $\bar{K}$.

(V, Θ) is non-singular if and only if its indecomposable summands are. If (V, Θ) is indecomposable, then (i) to (iii) are equivalent to

(iv) There is a finite separable field extension L/K and an indecomposable non-singular r-form (U, Φ) over L with center L such that $(V, \Theta) \cong (U, tr_{L/K}\Phi)$.

Proof: (i)⇔(ii) is stated in ([9], Chap. 13, 1.D).
(ii)⇔(iii) follows from Lemma 2.2(ii).
(ii)⇔(iv) is proved in ([10], Prop. 4.5). □

We want to compare the discriminant to the invariants studied before. Let per be the permanent of separable r-forms defined in 4.14. If r is even, let det be the determinant of even degree r-forms defined in Lemma 6.2.

7.4 Theorem. (Discriminant of r-Forms)
The discriminant of r-forms induces a map $\Delta_r : \hat{W}_r(K) \to K/K^{*r}$ having the following properties:

(i) Let Θ, Ψ be r-forms over K with $\dim(\Theta) = n$, $\dim(\Psi) = m$. Then

$$\Delta_r(\Theta \oplus \Psi) = \Delta_r(\Theta)^{(-1)^m} \cdot \Delta_r(\Psi)^{(-1)^n} \in K/K^{*r}.$$

(ii) Let $r \geq 3$. There are r-forms Θ, Ψ over K such that

$$\Delta(\Theta) \cdot \Delta(\Psi) \neq 0 \ , \ \Delta(\Theta \otimes \Psi) = 0.$$

(iii) Let Θ be a separable r-form of dimension n over K. Then

$$\Delta_r(\Theta) = \left\{ \begin{array}{c} \det(\Theta)^{(r-1)^{n-1}} \\ \operatorname{per}(\Theta)^{(r-1)^{n-1}} \end{array} \right\} \in K^*/K^{*r} \textit{ if } r \textit{ is } \left\{ \begin{array}{c} \textit{even.} \\ \textit{odd} \end{array} \right\}.$$

(iv) Let r be even. There are r-forms Θ over K such that

$$\Delta_r(\Theta) \neq 0 \ , \ \det(\Theta) = 0.$$

Proof: By Lemma 7.2(ii), the class of $\Delta_{r,n}$ in K/K^{*r} is invariant for isomorphism classes of r-forms. Hence our map Δ_r is well defined.
(i) follows from Lemma 7.2(iv).
(ii) An example was given in the proof of Lemma 2.4(iii).
(iii) Using (i), it suffices to show this for an indecomposable separable r-form $(L, \mathrm{tr}_{\mathrm{L/K}}\langle b\rangle_r)$. Things are obvious for $n = 1$, so let $n \geq 2$. In Lemma 4.7, we defined the matrix $B \in \mathrm{GL}_n(\bar{K})$ and showed that $(L, \mathrm{tr}_{\mathrm{L/K}}\langle b\rangle_r) = \langle 1, \ldots, 1\rangle_r \circ B$ over $\bar{K}$. Hence we have $\Delta_{r,n}(L, \mathrm{tr}_{\mathrm{L/K}}\langle b\rangle_r) = \det(B)^{r(r-1)^{n-1}}$ by Lemma 7.2(ii).
Let r be even. In Lemma 4.16 we showed $\det(L, \mathrm{tr}_{\mathrm{L/K}}\langle b\rangle_r) = \det(B)^r \in K^*/K^{*r}$, hence $\Delta_{r,n}(L, \mathrm{tr}_{\mathrm{L/K}}\langle b\rangle_r) = \det(L, \mathrm{tr}_{\mathrm{L/K}}\langle b\rangle_r)^{(r-1)^{n-1}}$.
Let r be odd. Then $\mathrm{sgn}^{(r-1)^{n-1}}$ vanishes, hence $\mathrm{per}^{(r-1)^{n-1}} = \det^{(r-1)^{n-1}}$ on the wreath product. Therefore $\mathrm{per}(L, \mathrm{tr}_{\mathrm{L/K}}\langle b\rangle_r)^{(r-1)^{n-1}} \in K^*/K^{*r}$ is given by $\det(B)^{r(r-1)^{n-1}} = \Delta_{r,n}(L, \mathrm{tr}_{\mathrm{L/K}}\langle b\rangle_r)$. (iv) Examples are given in Lemma 7.5. □

7.5 Lemma. (Discriminant and Determinant of Hyperelliptic Curves) *Let $f \in K[x]$ be a monic polynomial of degree r and let $F \in K[x, y, z]$ be the homogenization of the polynomial $y^{r-1} + f(x)$. Then*

(i) $\Delta_{r,3}(F) = \Delta_{r,2}(f)^{r-2}$.

(ii) If r is even, then $\det(F) = 0$.

Proof: (i) More generally, let $f = b_0 + b_1 x + \cdots + b_r x^r$ and let $F(x, y, z) = b_\infty y^{r-1} z + \sum_i b_i x^i z^{r-i}$. We show that F is non-singular if and only if b_∞ and b_r are non-zero, and f is non-singular. Let $b_\infty = 0$. Then F has a singularity at (0,1,0). Let $b_r = 0$. Then $F(x, y, 0) = \frac{\partial F}{\partial x}(x, y, 0) = \frac{\partial F}{\partial y}(x, y, 0) = 0$, and $\frac{\partial F}{\partial z}(x, y, 0) = b_\infty y^{r-1} + b_{r-1} x^{r-1}$ has a non-trivial zero in $\bar{K}$. Hence F is singular. Now let $b_\infty b_r \neq 0$. Then F has exactly one zero $(0, 1, 0)$ and no singularity at ∞, and F is singular at $(x, y, 1)$ if and only if $y = 0$ and f is singular at x.
By the Nullstellensatz, it follows that $\Delta_{r,3}(F) = c b_\infty^\lambda b_r^\mu \Delta_{r,2}(f)^\nu$ with $c \in K$ and $\lambda, \mu, \nu \in \mathbb{N}$. Now consider $f = b_r x^r + b_0$ and $F = b_\infty y^{r-1} z + b_r x^r + b_0 z^r$. The polynomial f is singular if and only if $b_r b_0 = 0$, hence $\Delta_{r,2}$ must contain a monomial $b_r^{\nu_1} b_0^{\nu_2}$. Then $r\nu_1 = r\nu_2 = r(r-1)$ by Lemma 7.2(iii), hence $\Delta_{r,2}(f) = b_r^{r-1} b_0^{r-1}$. Now $\Delta_{r,3}$ contains the monomial $b_\infty^\lambda b_r^{\mu+\nu(r-1)} b_0^{\nu(r-1)}$, and by 7.2(iii) again, we have $r(\mu + \nu(r-1)) = (r-1)\lambda = \lambda + r\nu(r-1) = r(r-1)^2$. Hence $\lambda = r(r-1)$, $\mu = r - 1$, and $\nu = r - 2$.
(ii) By Theorem 6.4, det can be computed using the generalized Leibniz formula

$$\det(F) = \sum_{\sigma_2, \ldots, \sigma_r \in S_3} \mathrm{sgn}(\sigma_2 \cdots \sigma_r) \prod_{i=1}^{3} a(i, \sigma_2 i, \ldots, \sigma_r i),$$

where $(x_1, x_2, x_3) = (x, y, z)$ and $a(i_1, \ldots, i_r)$ is a multiple of the coefficient for the monomial $x_{i_1} \cdots x_{i_r}$ in F (cf. Lemma 2.1). For each monomial $\prod_i a(i, \sigma_2 i, \ldots, \sigma_r i)$ in det, the index 2 occurs r times in the arguments of the factors all together. But in all the monomials of F, the variable $y = x_2$ occurs only $r - 1$ times in the monomial $y^{r-1} z$, hence at least one factor in each monomial of det must vanish. □

The Hyperdeterminant

Let $V = K^n$ and let $X = \mathbb{P}(V) \times \cdots \times \mathbb{P}(V) \hookrightarrow \mathbb{P}(V^{\otimes r})$ be the Segre embedding. The space $(V^{\otimes r})^*$ is identified with the space of (not necessarily symmetric) multilinear r-forms

$$f = f(x_1^{(1)}, \ldots, x_n^{(1)}, \ldots, x_1^{(r)}, \ldots, x_n^{(r)}) = \sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} a_{i_1 \ldots i_r} x_{i_1}^{(1)} \cdots x_{i_r}^{(r)}.$$

The dual variety $X^\vee \subset \mathbb{P}(V^{\otimes r})^*$ consists of the forms f for which the hypersurface $(f = 0) \subset \mathbb{P}(V^{\otimes r})$ is singular. Let $\mathrm{hdet}_{r,n} := \Delta_X$ be the X-discriminant, also called the hyperdeterminant of format $n \times \cdots \times n$ (cf. [9], Chap. 14, 1.A).

7.6 Lemma. *Let $r \geq 2$, let (V, Θ) be an r-form, and let $f \in K[x_i^{(j)}]$ be the multilinear r-form given by*

$$f(x_1^{(1)}, \ldots, x_n^{(1)}, \ldots, x_1^{(r)}, \ldots, x_n^{(r)}) := \Theta(\sum_{i=1}^{n} x_i^{(1)} v_i, \ldots, \sum_{i=1}^{n} x_i^{(r)} v_i).$$

(i) The hyperdeterminant $\mathrm{hdet}_{r,n}(f)$ *is a homogeneous polynomial expression in the coefficients of f.*

(ii) The hyperdeterminant $\mathrm{hdet}_{r,n}(f)$ *vanishes if and only if there are non-zero vectors $x^{(1)}, \ldots, x^{(r)} \in \bar{K}^n$ such that*

$$f(x) = \tfrac{df}{dx_i^{(j)}}(x) = 0$$

for every $i = 1, \ldots, n$ and $j = 1, \ldots, r$.

(iii) The hyperdeterminant $\mathrm{hdet}_{r,n}(f)$ *vanishes if and only if there are non-zero vectors $v^{(1)}, \ldots, v^{(r)} \in V \otimes_K Kbar$ such that*

$$\Theta(v^{(1)}, \ldots, v^{(j-1)}, u, v^{(j+1)}, \ldots, v^{(r)}) = 0$$

for every $u \in V \otimes_K Kbar$ and $j = 1, \ldots, r$.

Proof: (i): This is found in ([9], Chap. 14, Th. 1.3). (ii) and (iii) follow from ([9], Chap. 14, Prop.1.1). □

7.7 Lemma. (Hyperdeterminant of r-Forms)

(i) Let f be an r-form with $\Delta_{r,n}(f) = 0$. Then $\mathrm{hdet}_{r,n}(f) = 0$.

(ii) For bilinear forms, the hyperdeterminant is equal to the discriminant.

(iii) For cubic forms of dimension 2, the hyperdeterminant is equal to the discriminant.

(iv) For a diagonal r-form of dimension n, the hyperdeterminant vanishes if $r = 3$ and $n \geq 3$ or $r \geq 4$ and $n \geq 2$.

Proof: (i) By Lemma 7.3(iii) and the remark after Definition 1.2, there is a vector $v \in V \otimes_K \bar{K}$ such that $\Theta_{v,\ldots,v} = 0 \in V^*$. Then $v^{(1)} = \cdots = v^{(r)} := v$ satisfies condition for the vanishing of the hyperdeterminant in Lemma 7.6(iii).

Let The negation of 7.3(iii) implies 7.6(iii).

(ii): [9], Chap. 14, Prop. 1.1.

(iii) This follows from the formulas for $\Delta_{3,2}$ and in $\mathrm{hdet}_{3,2}$ given in ([9], Chap. 12, (1.34)) and ([9], Chap. 14, Prop. 1.7). Note that, in the notation used there, we have

$$a_0 = \binom{3}{3,0} a_{000} = a_{000} \ , \ a_1 = \binom{3}{2,1} a_{001} = 3a_{001} = 3a_{010} = 3a_{100},$$
$$a_2 = \binom{3}{1,2} a_{011} = 3a_{011} = 3a_{101} = 3a_{110} \ , \ a_3 = \binom{3}{0,3} a_{111} = a_{111}$$

(cf. Lemma 2.1), and that the normalization of the two formulas differs by the factor 27 .

(iv) We use the condition in Lemma 7.6(iii). Let $e_i \in \bar{K}^n$ denote the i-th unit vector. In the case $r = 3$ and $n \geq 3$, let $v^{(i)} := e_i$ $(i = 1, 2, 3)$. In the case $r \geq 4$, let $v^{(1)} = v^{(2)} := e_1$ and $v^{(3)} = \cdots = v^{(n)} = e_2$. Then $\mathrm{hdet}_{r,n}(f)$ vanishes by Lemma 7.6(iii). $\square$

8 Zeta Functions of Separable r-Forms over Finite Fields

This section is motivated by the study of motives corresponding to varieties over a field. In the theory of quadratic forms, recently the motives corresponding to the induced varieties have become an object of study, and therefore it seems appropriate to ask whether the determinant of an r-form just depends on the corresponding motive.
Let K be a finite field, and let $r > 2$. Let (V, Θ) be an r-form of dimension n over K, and let $X \subset \mathbb{P}^{n-1}_K$ be the projective hypersurface described by $\Theta = 0$. A well-studied invariant of the motive corresponding to X is the zeta function, and the Tate conjecture implies that it determines the motive. Thus, if we assume that the determinant of r-forms gives an invariant for the induced motives over k, then we would expect that r-forms with equal zeta function should have equal determinant. However, the following argument shows that we can not not expect too much: The zeta function is a projective invariant, hence it remains unchanged if we exchange Θ by a multiple $a\Theta$ with $a \in K^*$. But this changes the determinant by the factor a^n, hence its class in K^*/K^{*r} is changed if the dimension n is not a multiple of the degree r.

We want to compare determinant an zeta function of Θ. The zeta function is defined as

$$\zeta(\Theta, t) = \zeta(X, t) := exp(\sum_{i \geq 1} \frac{\nu_i}{i} t^i) \in \mathbb{Q}[[t]],$$

where $\nu_i := \mathrm{card}(X(\mathbb{F}_{q^i}))$ is the number of $\mathbb{F}_{q^i}$-rational points of X. If (V, Θ) is non-singular, then the Weil conjectures, proved by Dwork and Deligne, imply that $\zeta(\Theta, t)$ is a rational function, and that it can be computed in terms of étale cohomology using the Lefschetz Trace Formula:

8.1 Lemma. *Let (V, Θ) be a non-singular r-form of dimension n over K, and let $X \subset \mathbb{P}^{n-1}_K$ be the projective hypersurface described by Θ. Let $\bar{K}$ be a separable closure of K, let $\bar{X} := X \times_K \bar{K}$, and let $F_X : \bar{X} \to \bar{X}$ be the geometric Frobenius. Let $l \neq \mathrm{char}(K)$ be a prime, and let $Q(\Theta, t) := \det(1 - F_X^* t | H^{n-2}_{\acute{e}t}(\bar{X}, \mathbb{Q}_l)) \in \bar{\mathbb{Q}}[t]$. Then $Q(\Theta, t)$ has integer coefficients, and the zeta function of Θ is given as*

$$\zeta(\Theta, t) = Q(\Theta, t)^{(-1)^{n-1}} \prod_{i=1,\ldots,n-2,\ i \neq \frac{n-2}{2}} \frac{1}{1 - q^i t} \in \mathbb{Q}(t).$$

Proof: This follows from the Weil conjectures (cf. [24], Theorem VI.12.4) and the description of the groups $H^i_{\acute{e}t}(\bar{X}, \mathbb{Q}_l))$ in ([6], Theorem 1.6). $\square$

Now let (V, Θ) be a separable r-form. In particular, this means that X is non-singular, and the formula from the theorem holds. Let $Y \subset \mathbb{P}^{n-1}_K$ denote the Fermat hypersurface of degree r, given by the diagonal r-form $(\bar{K}^n, \langle 1, \ldots, 1 \rangle_r)$.

In Lemma 3.8(v), we showed that (V,Θ) and $(K^n,\langle 1,\dots,1\rangle_r)$ are isomorphic over $\bar{K}$, or, in other words, that (V,Θ) is a form of the Fermat form. In particular, this means $\bar{X}=\bar{Y}$. In Theorem 4.6, we identified separable r-forms of dimension n over K with the elements of the cohomology set $H^1(K, S_n \int \mu_r)$.

We fix some notation: Let K be a finite field which contains the r-th roots of unity, i.e. $K=\mathbb{F}_q$ such that $q\equiv 1 \bmod r$. Let $K=\mathbb{F}_q$ be a finite field. For $n\in\mathbb{N}$, let $b_n\in\mathbb{F}_{q^n}^*$ be a generator of the multiplicative group in the n-th extension field of K. We may choose the family of generators $\{b_n \mid n\in\mathbb{N}\}$ as a projective system with respect to the norms, i.e. such that $N_{\mathbb{F}_{q^m}/\mathbb{F}_{q^n}}(b_m)=b_m^{\frac{q^m-1}{q^n-1}}=b_n$ for all $m,n\in\mathbb{N}$ with $n|m$. By Lemma 3.8, every indecomposable separable r-form over K is isomorphic to $(\mathbb{F}_{q^n}, \mathrm{tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}\langle b_n^i\rangle_r)$ for some n and i. Let $\zeta=\zeta_r$ denote the primitive r-th root of unity $b_1^{1-q/r}\in K$, and let σ_n denote the transitive cycle $(1..n)\in S_n$.

8.2 Lemma. *The cohomology class in $H^1(\mathbb{F}_q, S_n\int\mu_r)$ corresponding to the r-form $(\mathbb{F}_{q^n}, \mathrm{tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}\langle b_n^i\rangle_r)$ contains a 1-cocycle $z\in Z^1(\mathbb{F}_q, S_n\int\mu_r)$ such that for the arithmetic Frobenius $f=f_q\in G_{\mathbb{F}_q}$ we have*

$$z_f=(\sigma_n,(\zeta^i,1,\dots,1))\in S_n\textstyle\int\mu_r.$$

Proof: This cocycle is computed in Lemma 4.7: In the notation used there, we choose $b:=b_n^i$, $\beta\in\overline{\mathbb{F}}_q$ an r-th root of b, and $t_j:=f^{j-1}$ for $j=1,\dots,n$. Then $\rho_f=\sigma_n\in S_n$ and

$$\frac{t_j(\beta)}{ft_{\rho_f^{-1}j}(\beta)}=\frac{f^{j-1}(\beta)}{f^{\sigma_n^{-1}j}(\beta)}=\left\{\begin{array}{ll}\beta^{1-q^n}=\zeta^i & \text{for } j=1\\ 1 & \text{for } j>1\end{array}\right\}.$$

□

Let $\vartheta:=z_f=(\sigma_n,(\zeta^i,1,\dots,1))$. We introduce some more notation: Let $\mu_r\subset K$ be the group of r-th roots of unity, let $A:=\mu_r^n/\mu_r$ be the quotient by the diagonal embedding $\mu_r\hookrightarrow\mu_r^n$ and let $\check{A}:=\{a=(a_1,\dots,a_n)\in(\mathbb{Z}/r)^n \mid \Sigma_i a_i=0\}$. Then $\check{A}\cong\mathrm{Hom}(A,\mu_r)$ by the pairing

$$\check{A}\times A\to\mu_r\ ,\ a(\bar{\alpha}):=\textstyle\prod_i\alpha_i^{a_i}\ \text{for}\ \alpha=(\alpha_1,\dots,\alpha_n)\in\mu_r^{\oplus n}.$$

The group $A\subset S_n\int\mu_r/\mu_r=\mathrm{Aut}(\bar{Y})$ (cf. Lemma 4.5) acts on $H^{n-2}_{\acute{e}t}(\bar{Y},\mathbb{Q}_l)$ by functoriality. For $a\in\check{A}$, let $V_a:=\{v\in H^{n-2}_{\acute{e}t}(\bar{Y},\mathbb{Q}_l)\mid\alpha^*v=a(\alpha)\cdot v \text{ for all } \alpha\in A\}$. Let $A_n^r:=\{a\in\check{A}\mid a_i\neq 0 \text{ for } i=1,\dots,n\}$.

Now let $L=\mathbb{Q}(\mu_r)$ be the r-th cyclotomic field, and fix an embedding $L\subset\mathbb{C}$. Let f be the order of $p=\mathrm{char}(K)$ in $G:=G_{L/\mathbb{Q}}\cong(Z/r)^*$, let $H\subset G$ be the subgroup generated by p. Let $q_0:=p^f$. Then $\mathbb{F}_{q_0}$ is the smallest field of characteristic p containing the r-th roots of unity and we have $q=q_0^m$ for some $m\in\mathbb{N}$.

8.3 Lemma. *Let $\mathfrak{p}$ be a prime ideal in L lying over p. Then*

(i) The residue field and the decomposition group of $\mathfrak{p}$ are $\kappa(\mathfrak{p}) \cong \mathbb{F}_{q_0}$, $G_\mathfrak{p} = H$.

(ii) Identifying $\kappa(\mathfrak{p}) = \mathbb{F}_{q_0}$ by (i), there is a unique character $\chi = \chi_q : \mathbb{F}_q^* \to \mathbb{C}$ of exact order r such that $\chi_q(u) \equiv u^{q-1/r}$ mod $\mathfrak{p}$ for $u \in \mathbb{F}_q$.

Proof: (i) Since $(p, r) = 1$, p is unramified in L, $G_\mathfrak{p} \cong G_{\kappa(\mathfrak{p})/\mathbb{F}_p}$ is generated by p, thus equal to H.
(ii): For $u \in \mathbb{F}_q$, $u^{q-1/r}$ is an r-th root of unity. The roots of $x^r - 1$ in L are distinct modulo $\mathfrak{p}$, hence $\chi(u) \in L$ is uniquely defined. □

8.4 Definition. *For $a \in A_n^r$, the Jacobi sum is defined as*

$$J(a) = J_q(a) := (-1)^n \prod_{\substack{u_2, \dots, u_n \in K^* \\ u_2 + \dots + u_n = -1}} \chi(u_2)^{a_2} \cdots \chi(u_n)^{a_n} \in L.$$

The following lemma gives a formula for the zeta function of separable r-forms over $\mathbb{F}_q$, which was proved by Brünjes in [2]. Our formula is slightly different from the one given by Brünjes, but the proof follows his exposition:

8.5 Lemma. *Let A_n^r/σ denote the set of orbits $[a] \subset A_n^r$ under the action of $\mathbb{Z}$ via $\mathbb{Z} \to S_n$, $m \mapsto \sigma^m$, and for $a \in A_n^r$, let $l(a) := \operatorname{card}([a])$ denote the length of its orbit. Then*

$$Q(\Theta, t) = Q_{(0,\dots,0)}(t) \cdot \prod_{[a] \in A_n^r/\sigma} Q_{[a]}(\Theta, t),$$

where

$$Q_{(0,\dots,0)}(t) := \left\{ \begin{array}{cc} 1 - q^{-\frac{n-2}{2}} & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{array} \right\} \text{ and}$$
$$Q_{[a]}(\Theta, t) := 1 - \left(\prod_{i=0}^{l(a)-1} \sigma^i a(\alpha) \right) (J(a) \operatorname{sgn}(\sigma)\, t)^{l(a)}.$$

For the proof, we need some lemmas:

8.6 Lemma. *For the action of the geometric Frobenius morphisms F_X and F_Y on $H_{\acute{e}t}^{n-2}(\bar{X}, \mathbb{Q}_l) = H_{\acute{e}t}^{n-2}(\bar{X}, \mathbb{Q}_l)$, we have $F_X^* = \vartheta^* \circ F_Y^*$.*

Proof: This is proved in ([2], 3.12 (16)). □

8.7 Lemma. *$V = H^{n-2}_{\acute{e}t}(\bar{Y}, \mathbb{Q}_l)$ has a decomposition of $\mathbb{Q}_l[G_K]$-representations*

$$V = V_{(0,\dots,0)} \oplus \bigoplus_{a \in A^r_n} V_a.$$

We have $\dim_{\mathbb{Q}_l}(V_{(0,\dots,0)}) = \left\{ \begin{array}{ll} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{array} \right\}$ and $\dim_{\mathbb{Q}_l}(V_a) = 1$ for $a \in A^r_n$. The action of the geometric Frobenius F_Y respects this decomposition. It acts on $V_{(0,\dots,0)}$ with the eigenvalue $q^{-\frac{n-2}{2}}$, if n is even, and on V_a with the eigenvalue $J_q(a)$ for $a \in A^r_n$.

Proof: This is proved by Deligne in ([7], Prop. 7.11). □

8.8 Lemma. *There is a basis $\{v_a \,|\, a \in A^r_n\}$ of $\bigoplus_{a \in A^r_n} V_a$ such that $v_a \in V_a$ and*

$$\vartheta^* v_a = \sigma^{-1} a(\alpha) \mathrm{sgn}(\sigma) v_{\sigma^{-1}a}$$

for each $a \in A^r_n$. ϑ acts trivially on $V_{(0,\dots,0)}$.

Proof: This is proved in ([2], 9.17). □

Proof of Lemma 8.5: Putting the lemmas together, we see that the action of F_X on $H^{n-2}_{\acute{e}t}(\bar{X}, \mathbb{Q}_l)$ respects the subspaces $V_{(0,\dots,0)}$ and $V_{[a]} := \bigoplus_{b \in [a]} V_b$ for $a \in A^r_n$, so that we get

$$Q(\Theta, t) = Q_{(0,\dots,0)}(t) \cdot \prod_{[a] \in A^r_n/\sigma} Q_{[a]}(\Theta, t),$$

where

$$Q_{(0,\dots,0)}(t) := \det(1 - F^*_X \,|\, V_{(0,\dots,0)}) = \left\{ \begin{array}{ll} 1 - q^{-\frac{n-2}{2}} & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{array} \right\},$$
$$Q_{[a]}(\Theta, t) := \det(1 - F^*_X \,|\, V_{[a]}) \text{ for } a \in A^r_n.$$

Let $a \in A^r_n$ and let $l := l(a)$. By 8.6, we may choose a basis $\{v_{\sigma a}, v_{\sigma^2 a}, \dots, v_{\sigma^l a}\}$ of $V_{[a]}$ such that $v_{\sigma^i a} \in V_{\sigma^i a}$ and $\vartheta^* v_{\sigma^i a} = \sigma^{i-1} a(\alpha) \mathrm{sgn}(\sigma) v_{\sigma^{-1}a}$ for $i = 1, \dots, l$. Then we know from the Lemmas that $F^*_X \,|\, V_{[a]}$ is given by the matrix $(M_{[a]} \in M_l(L))$, where

$$M_{[a]} = \begin{pmatrix} 0 & x_1 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & & \ddots & x_{l-1} \\ x_l & 0 & \cdots & 0 \end{pmatrix} \text{ for } l > 1,\ M_{[a]} = (x_1) \text{ for } l = 1$$

with $x_i = \sigma^{i-1} a(\alpha) \mathrm{sgn}(\sigma) J(a)$.

Expanding the matrix $1 - M_{[a]}t$ along the first column, one checks that

$$Q_{[a]}(\Theta, t) = \det(1 - M_{[a]}t) = 1 - \prod_{i=1}^{l} x_i t = 1 - \left(\prod_{i=0}^{l-1} \sigma^i a(\alpha)\right) (J(a)\,\mathrm{sgn}(\sigma)\,t)^l.$$

□

In order to analyze the formula in Lemma 8.5, we want to know the prime decomposition of the Jacobi sum $J_q(a)$ in $L = \mathbb{Q}(\mu_r)$. This can be computed with Stickelberger's Theorem. We identify the primitive r-th root of unity $\zeta \in K^*$ with its image $\chi(\zeta) \in L$, and let the isomorphism $(\mathbb{Z}/r)^* \cong G = G_{L/\mathbb{Q}}$ be given by $t \mapsto s_t$, where $s_t(\zeta) := \zeta^t$. In Lemma 8.3, we saw that the decomposition group of $\mathfrak{p}$ is the subgroup $H \subset G$ generated by p, which has order f. Let $t_1, \dots, t_s \in (\mathbb{Z}/r)^*$ be representatives for the classes in G/H and let $\mathfrak{p}_i := \sigma_{-t_i}^{-1}\mathfrak{p}(i = 1, \dots, s)$ be the different prime ideals in L over p. For $x \in \mathbb{Q}$, let $\langle x \rangle \in \mathbb{Q}$ be the fractional part of x, i.e. the unique number with $0 \leq \langle x \rangle < 1$ and $\langle x \rangle - x \in \mathbb{Z}$.

The formula in the next lemma is found in ([33], (8)).

8.9 Lemma. *Let $q = q_0^m$. Then the prime decomposition of $J_q(a)$ in L is*

$$(J_q(a)) = \mathfrak{p}_1^{c_1(a)} \cdots \mathfrak{p}_s^{c_s(a)} \text{ with } c_\nu(a) := m \sum_{j=0}^{f-1} (\sum_{i=1}^{n} \langle \frac{t_\nu p^j a_i}{r} \rangle - 1) \in \mathbb{Z}.$$

Proof: First, let $K = \mathbb{F}_{q_0}$. Let $\zeta_p \in \mathbb{C}$ be a primitive p-th root of unity and let $\mathfrak{P}$ be the prime ideal in $L(\zeta_p)$ with $\mathfrak{p} = \mathfrak{P}^{p-1}$. Let $\lambda : K^* \to \mu_p$ be the additive character $\lambda(u) := \zeta_p^{tr_{K/\mathbb{F}_p}(u)}$ and let $g(a_i) = g(\chi^{a_i}) := \sum_{u \in K^*} \chi(u)^{a_i} \lambda(u)$ be the Gauss sum for χ^{a_i}. André Weil shows in ([32], p501) that $J_q(a) = \frac{1}{q} g(a_1) \cdots g(a_n) = N_{K/\mathbb{F}_p}(\mathfrak{p})^{-1} g(a_1) \cdots g(a_n)$ and Stickelberger's Theorem (cf. [23], Th. 1.2.2) gives

$$(g(a_i)) = \mathfrak{P}^{(p-1)\Theta(a_i)} \text{ in } L(\zeta_p),$$

with the Stickelberger element $\Theta(a_i) := \sum_{t \in G} \langle \frac{ta_i}{r} \rangle s_{-t}^{-1} \in \mathbb{Q}[G]$. Thus,

$$(J_q(a)) = \mathfrak{P}^{(p-1)\omega(a)} = \mathfrak{p}^{\omega(a)},$$

with $\omega(a) := \sum_{i=1}^{n} \Theta(a_i) + \sum_{t \in G} s_t = \sum_{t \in G} (\sum_{i=1}^{n} \langle \frac{ta_i}{r} \rangle - 1) s_{-t}^{-1} \in \mathbb{Z}[G]$. Collecting the coefficients in each coset of H yields the Lemma for $m = 1$. For $q = q_0^m$, we have $J_q(a) = J_{q_0}(a)^m$ by Lemma 8.7, since $J_q(a)$ is the eigenvalue on V_a for the geometric Frobenius on $Y/\mathbb{F}_q$. □

8.10 Lemma. *Assume that the prime field $\mathbb{F}_p$ already contains the r-th roots of unity, i.e. $p \equiv 1 \bmod r$, and that n is a multiple of r. Then $(1, \dots, 1) \in A_n^r$ and for any $a \in A_n^r$, the Jacobi sums $J(a)$ and $J(1, \dots, 1)$ generate the same ideal in*

L if and only if $a = (1, \ldots, 1)$.

Proof: We compare the prime decomposition for the Jacobi sums given in Lemma 8.9: Since $p \equiv 1 \bmod r$, the decomposition group of $\mathfrak{p}$ vanishes, and we have $\varphi(r)$ distinct primes $\{\mathfrak{p}_t \,|\, t \in G\}$ in L lying over p. Let $a \in A : n^r$. Then $a_i \not\equiv 0 \bmod r$ for $i = 1, \ldots, n$ and therefore $c_1(a) = m \sum\limits_{i=1}^{n} \langle \frac{a_i}{r} \rangle - 1 \geq m \sum\limits_{i=1}^{n} \frac{1}{r} - 1 = c_1(1, \ldots, 1)$ and equality holds if and only if $a = (1, \ldots, 1)$. $\square$

8.11 Theorem. *Let K be a finite field of characteristic p, whose prime field $\mathbb{F}_p$ contains the r-th roots of unity. Let n be a multiple of r and let (V, Θ) and (W, Ψ) be r-forms of dimension n over K having the same zeta function. Then* $\det(\Theta) = \det(\Psi)$*, where* $\det$ *is the determinant of separable r-forms from Definition 4.14.*

Proof: Let $z, y : G_K \to S_n \int \mu_r$ be the 1-cocycles from Lemma 8.2, corresponding to Θ and Ψ, and let $z_f = (\sigma, (\alpha_1, \ldots, \alpha_n))$, $y_f = (\tau, (\beta_1, \ldots, \beta_n))$. Since Θ and Ψ have the same zeta function, we have $\prod\limits_{[a] \in A_n^r/\sigma} Q_{[a]}(\Theta, t) = \prod\limits_{[a] \in A_n^r/\tau} Q_{[a]}(\Psi, t)$ by Lemmas 8.1 and 8.5. Since $Q_{[1,\ldots,1]}(\Theta, t) = 1 - \mathrm{sgn}(\sigma)(\prod_i \alpha_i) J_q(1, \ldots, 1)t = 1 - \det(z_f) J_q(1, \ldots, 1)t$, we know that $x_0 := (J_q(1, \ldots, 1)\det(z_f))^{-1}$ is a zero of $Q_{[a]}(\Psi, t)$ for some $a \in A_n^r$. But then it follows from the formula for $Q_{[a]}$, given in Lemma 8.5, that $J_q(a) = \xi J_q(1, \ldots, 1)$ with some root of unity ξ and therefore $a = (1, \ldots, 1)$ by Lemma 8.10. Now $Q_{[1,\ldots,1]}(\Psi, x_0) = 1 - \frac{det(y_f)}{det(z_f)} = 0$ and since f generates $G_K = \hat{\mathbb{Z}}$ topologically, the value $\det(z_f) = \det(\Theta)(f)$ at f determines the class of $\det(\Theta)$ in $H^1(K, \mu_r)$, so we have $\det(\Theta) = \det(\Psi)$. $\square$

9 Hyperbolic r-Forms and the Witt Ring

In the Witt-Grothendieck ring of quadratic forms, the additive subgroup H generated by the hyperbolic plane $h = \langle 1, -1 \rangle$ is an ideal, and the Witt ring of quadratic forms is defined as the factor ring $W(K) := \hat{W}(K)/H$. Let $\hat{I} := \ker(\hat{W} \to \mathbb{Z})$ denote the augmentation ideal and let $I := \ker(W(K) \to \mathbb{Z}/2)$ denote the fundamental ideal. Let det be the determinant map $\det : \hat{W}^{+}(K) \to K^*/K^{*2}$ to $\hat{W}_r(K)$. The discriminant of the quadratic form q is defined as $d(q) := (-1)^{\lfloor \frac{dim(q)}{2} \rfloor} \det(q)$. The discriminant induces a group isomorphism $e_1 : I/I^2 \cong K^*/K^{*2}$.

In order to give a definition for a Witt ring of r-forms, we want to find degree r analogues of the discriminant and the hyperbolic ideal in the Witt-Grothendieck ring of r-forms. The following lemma lists the minimal requirements one would expect from such a pair:

9.1 Lemma. *Let $r > 2$ and let K be a field such that $r!$ is invertible. Let $d : \hat{W}_r^{sep}(K) \to K^*/K^{*r}$ be the permanent or the determinant, where the determinant is available only if$r$ is even. Let $H \subset \hat{W}_r^{sep}(K)$ be an ideal such that* $\dim(H) \equiv 0$ *modulo r and $d(H) = 1$, and let $W_r(K) := \hat{W}_r^{sep}(K)/H$. Let $\hat{I}_r \subset \hat{W}_r^{sep}(K)$ denote the kernel of the dimension map* $\dim : \hat{W}_r^{sep}(K) \to \mathbb{Z}$, *and let $I_r \subset W_r(K)$ denote the kernel of the dimension index* $\dim : W_r(K) \to \mathbb{Z}/r$. *Then there is a commutative diagram of abelian groups with short exact rows and columns*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & H_0 \arrow[r] \arrow[d] & \hat{I}_r \arrow[r] \arrow[d] & I_r \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H \arrow[r] \arrow[d] & \hat{W}_r(K) \arrow[r] \arrow[d] & W_r(K) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & r\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/r \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

and d induces a surjective group homomorphism

$$d : I_r/I_r^2 \to K^*/K^{*r}.$$

Proof: The diagram is clear from the definition. We have $d(H) = 1$, so that d is well defined on I_r, and the product formula in 4.15 shows $d(I_r^2) = 1$. The induced map is clearly surjective. □

Having in mind the situation for quadratic forms, we would expect that this map is an isomorphism for the right choice of d and H. Up to now, there is no good proposal for H, but following an idea of Carlsson given in [4], we propose the following degree r analogue of the hyperbolic plane $h_2 = \langle 1, -1\rangle$, which should be contained in H:

9.2 Definition. *Let $r \neq 2$ be a prime and let $\phi(x) := x^{r-1} + \cdots + x + 1 \in K[x]$ be the r-th cyclotomic polynomial. Since $r! \neq 0$ in K, the polynomial ϕ is separable, and $L := K[x]/(\phi)$ is a separable K-algebra of dimension $r-1$ over K. We define*

$$h_r := \langle 1\rangle_r \oplus (L, tr_{L/K}\langle x\rangle_r) \in \hat{W}_r^{sep}(K).$$

9.3 Lemma.

(i) The r-form h_r has dimension r and permanent 1.

(ii) Every element in K occurs as a value of the homogeneous r-form h_r.

(iii) If K contains a primitive r-th root of unity ζ, then $h_r \cong \langle 1, \zeta, \ldots, \zeta^{r-1}\rangle_r$.

Proof: (i) It is clear that $\dim(h_r) = r$. By Lemma 4.16, we have

$$d(h_r) = N_{L/K}(x) = \prod_{i=1}^{r-1} x^i = 1 \in K^*/K^{*r}.$$

(ii) Let $a \in K$. Then $\mathrm{tr}_{L/K}(x(\frac{a}{r^2} + x)^r) = \sum_{k=0}^{r} \binom{r}{k}(\frac{a}{r^2})^k \,\mathrm{tr}_{L/K}(x^{r-k+1})$. For $i \geq 0$ we have

$$\mathrm{tr}_{L/K}(x^i) = x^i + \ldots x^{(r-1)i} = \left\{ \begin{array}{ll} r-1 & \text{if } i \equiv 0 \bmod r \\ -1 & \text{else} \end{array} \right\}.$$

Together this gives $\mathrm{tr}_{L/K}(x(\frac{a}{r^2} + x)^r) = r \cdot \binom{r}{1}\frac{a}{r^2} - \sum_{k=0}^{r}\binom{r}{k}(\frac{a}{r^2})^k = a - (\frac{a}{r^2}+1)^r$.

Now let $v := (\frac{a}{r^2} + 1, \frac{a}{r^2} + x) \in K \oplus L$. Then

$$h_r(v, \ldots, v) = (\tfrac{a}{r^2} + 1)^r + \mathrm{tr}_{L/K}(x(\tfrac{a}{r^2} + x)^r) = a.$$

(iii) If $\zeta \in K$ is a primitive r-th root of unity, then ϕ decomposes over K and we have an isomorphism of K-algebras $L \xrightarrow{\sim} K^{\oplus r-1}$, $l \mapsto (\sigma_1 l, \ldots, \sigma_{r-1} l)$, where $\sigma_i \in Hom_K(L, K)$ is given by $x \to \zeta^i$. One checks that this induces an isomorphism of r-spaces which proves the Lemma. □

In order to test this definition, we let H be the ideal generated by h_r and compute the group I/I^2 in the case that $K = \mathbb{F}_q$ is a finite field. The group K^*/K^{*r} is cyclic of order r if $q \equiv 1$ modulo r, i.e. if K contains the r-th roots of unity. If K does not contains the r-th roots of unity, then K^*/K^{*r} is trivial. The group I/I^2 that comes out from the Lemma, is not even finitely generated. This suggests that we may have to choose H much bigger than (h_r) in order to obtain the imagined results.

9.4 Notation. *Let $K = \mathbb{F}_q$ be a finite field. For $n \in \mathbb{N}$, let K_n denote the extension field $\mathbb{F}_{q^n}$ and let $b_n \in K_n^*$ be a generator of the multiplicative group. Here we choose the family of generators $\{b_n \mid n \in \mathbb{N}\}$ as a projective system with respect to the norms, i.e. such that $N_{K_m/K_n}(b_m) = b_m^{\frac{q^m-1}{q^n-1}} = b_n$ for all $m, n \in \mathbb{N}$ with $n|m$. For $n \in \mathbb{N}$ and $0 \le i < r$, let y_n^i denote the n-dimensional indecomposable separable r-form $(K_n, \mathrm{tr}_{K_n/K}\langle b_n^i\rangle_r) \in \hat{W}_r^{sep}(K)$, and let $z_n^i := y_n^i - n \cdot y_i^0 \in \hat{I}_r$. By 3.8, every indecomposable separable r-form over K is isomorphic to y_n^i for some n and i, so $\hat{W}_r(K)$ is generated by the y_n^i and $\hat{I}_r(K)$ is generated by the z_n^i $(n \in \mathbb{N}, 0 \le i < r)$.*

9.5 Lemma. (The Ring of Separable r-Forms over a Finite Field)
Let $r \neq 2$ be a prime and let $K = \mathbb{F}_q$ be a finite field. Let $H := (h_r)$.

(i) Let $q \equiv 1 \bmod r$, i.e. $\mathbb{F}_q$ contains r-th roots of unity. Then $\hat{W}_r^{sep}(\mathbb{F}_q)$ is a free $\mathbb{Z}$-module with basis $\{y_n^i \mid n \ge 1, 0 \le i < r\}$ and I/I^2 is a free $\mathbb{Z}/r$-module with basis $\{z_1^1\} \cup \{z_{r^\nu}^i \mid \nu \ge 1, 0 \le i < r\}$.

(ii) Let $q \not\equiv 1 \bmod r$, i.e. $\mathbb{F}_q$ does not contain r-th roots of unity. Then $\hat{W}_r^{sep}(\mathbb{F}_q)$ is a free $\mathbb{Z}$-module with basis $\{y_n^0 \mid n \ge 1\} \cup \{y_{(r-1)n}^1 \mid n \ge 1\}$ and I/I^2 is a free $\mathbb{Z}/r$-module with basis $\{z_{r^\nu}^0 \mid \nu \ge 1\} \cup \{z_{r^\nu(r-1)}^1 \mid \nu \ge 0\}$.

Proof: Every indecomposable separable r-form over $\mathbb{F}_q$ is equivalent to y_n^i for some n and i, hence $\hat{W}_r^{sep}(\mathbb{F}_q)$ is generated by the y_n^i and $\hat{I}_r$ is generated by the z_n^i. We have $I_r/I_r^2 \cong \hat{I}_r/(\hat{I}_r^2 + H_0) \cong \hat{I}_r/(\hat{I}_r^2 + h_r\hat{I}_r)$ and $h_r \equiv r y_1^0 \bmod \hat{I}_r$ implies $r\hat{I}_r \equiv h_r\hat{I}_r \bmod \hat{I}_r^2$ and thus $rI_r \subset I_r^2$. Therefore I_r/I_r^2 is a $\mathbb{Z}/r$-module given by generators z_n^i and relations coming from multiplication in $\hat{I}_r$. For any $n, m \in \mathbb{N}$ we have $K_n \cap K_m = K_{(n,m)}$, $K_n \cdot K_m = K_{[n,m]}$ and 1.10 gives product formulas

$$y_n^i \cdot y_m^j = \bigoplus_{\sigma \in G(K_{(n,m)}/K)} (K_{[n,m]}, \mathrm{tr}\langle a_n^i \cdot \sigma a_m^j\rangle) = \sum_{k=1}^{(m,n)} y_{[m,n]}^{i\frac{q^{[m,n]}-1}{q^n-1} + q^k j \frac{q^{[m,n]}-1}{q^m-1}}, \quad (1)$$

$$z_n^i \cdot z_m^j = \sum_{k=1}^{(m,n)} z_{[m,n]}^{i\frac{q^{[m,n]}-1}{q^n-1} + q^k j \frac{q^{[m,n]}-1}{q^m-1}} - nz_m^j - mz_n^i. \quad (2)$$

(i) Let $q \equiv 1 \bmod r$. For $n \in \mathbb{N}$, the group K_n^*/K_n^{*r} is cyclic of order r and the Frobenius automorphism operates trivially. Thus, by 3.8(ii), the equivalence classes of indecomposable separable r-forms of dimension n over $\mathbb{F}_q$ are $y_n^0, \ldots, y_n^{r-1}$. This establishes the set of generators for $\hat{W}_r^{sep}(\mathbb{F}_q)$. With $q \equiv 1$ we have $q^k \equiv 1$ and $\frac{q^{[m,n]}-1}{q^n-1} = \sum_{k=0}^{[m,n]/n-1} q^{kn} \equiv \frac{[m,n]}{n} \bmod r$, so (2) yields the following relations modulo I_r^2:

$$nz_m^j + mz_n^i \equiv (m,n)\, z_{[m,n]}^{i\frac{[m,n]}{n} + j\frac{[m,n]}{m}} \text{ for } n, m \in \mathbb{N}\ ,\ 0 \le i, j < r. \quad (3)$$

With $n = m = 1$ we get $z_1^i + z_1^j \equiv z_1^{i+j}$ and thus $z_1^i \equiv iz_1^1$ for $0 \le i < r$. With $m = 1$, $i = 0$ we get $z_n^0 + nz_1^j \equiv z_n^{jn}$ and $n = m$, $i = j = 0$ yields $nz_n^0 \equiv 0$. For $(n, r) = 1$, we get $z_n^0 \in (n, r)z_n^0 \equiv 0$ and there is n^{-1} with $nn^{-1} \equiv 1$, so we get

$$z_n^i \equiv z_n^{in^{-1}n} \equiv z_n^0 + nz_1^{in^{-1}} \equiv z_n^0 + iz_1^1 \equiv iz_1^1 \text{ for } (n, r) = 1\ ,\ 0 \le i < r. \quad (4)$$

Relation (3) with $m = r^\nu$, $\nu \geq 1$, $(r,n) = 1$ yields $z^{jn}_{r^\nu n} \equiv n z^j_{r^\nu} + r^\nu z^0_n \overset{(4)}{\equiv} n z^j_{r^\nu}$ and therefore

$$z^i_{r^\nu n} \equiv n z^{in^{-1}}_{r^\nu} \text{ for } (n,r) = 1\ ,\ \nu \geq 1\ ,\ 0 \leq i < r. \tag{5}$$

Relations (4) and (5) establish the generating set for I_r/I_r^2. Taking the free module on this basis and using (4) and (5) as definition for the other z^i_n one checks back that all relations (3) are valid, so that the modules are isomorphic.
(ii) Let $q \not\equiv 1 \bmod r$. For $n \in \mathbb{N}$, r divides $q^n - 1$ if and only if $r-1$ divides n, so $K^*_n/K^{*r}_n = \{1, a_n, \ldots, a^{r-1}_n\}$ is cyclic of order r if $r-1$ divides n and trivial else. If $r-1$ divides n, the Frobenius automorphism operates transitively on the set $\{a_1, \ldots, a^{r-1}_n\} \subset K^*_n/K^{*r}_n$, so by 3.8(ii) the equivalence classes of indecomposable separable r-forms of dimension n over $\mathbb{F}_q$ are y^0_n and y^1_n. This establishes the set of generators for $\hat{W}^{sep}_r(\mathbb{F}_q)$. Again, we can simplify the product formula (2) and have relations $n z^j_m + m z^i_n \equiv \sum_{k=0}^{(m,n)-1} z^{i\frac{[m,n]}{n} + jq^k \frac{[m,n]}{m}}_{[m,n]}$, since either $i = 0$, or $r-1$ divides n and then $q^n \equiv 1$ and $\frac{q^{[m,n]}-1}{q^n - 1} \equiv \frac{[m,n]}{n}$ mod r, same for j and m. In particular, we get the following relations

$$m z^0_n + n z^0_m \equiv (m,n) z^0_{[m,n]}, \tag{3'}$$

$$m z^1_n + n z^0_m \equiv (m,n) z^1_{[m,n]} \text{ for } (r,m) = 1. \tag{4'}$$

Relation $(3')$ with $m = n$, $i = j = 0$ yields $n z^0_n \equiv 0$, and therefore

$$z^0_n \equiv 0 \text{ for } (r,n) = 1, \tag{5'}$$

$$z^0_{r^\nu n} \overset{(3')}{\equiv} n z^0_{r^\nu} + r^\nu z^0_n \overset{(5')}{\equiv} n z^0_{r^\nu} \text{ for } (r,n) = 1,\ \nu \geq 1. \tag{6'}$$

Let $(r,n) = 1$ and let $n = n_1 n_2$ where n_1 is the maximal divisor of n prime to $r-1$. Then

$$(r-1) z^1_{r^\nu (r-1) n_2} \overset{(4')}{\equiv} (r-1) n_2 z^1_{r^\nu (r-1)} + r^\nu (r-1) z_{(r-1)n_2} \overset{(5')}{\equiv} (r-1) n_2 z^1_{r^\nu (r-1)}.$$

This implies $z^1_{r^\nu (r-1) n_2} \equiv n_2 z^1_{r^\nu (r-1)}$, and we get

$$\begin{aligned} z^1_{r^\nu (r-1) n} &\overset{(4')}{\equiv} n_1 z^1_{r^\nu (r-1) n_2} + r^\nu (r-1) n_2 z^0_{n_1} \\ &\overset{(5')}{\equiv} n_1 z^1_{r^\nu (r-1) n_2} \equiv n_1 n_2 z^1_{r^\nu (r-1)} = n z^1_{r^\nu (r-1)}. \end{aligned} \tag{7'}$$

Relations (5') to (7') establish the generating set for I_r/I_r^2. Taking the free module on this basis and using (5')-(7') as definition for the other z^i_n it is a tedious task to check back that all the relations induced by (2) are valid. This shows that the modules are isomorphic. □

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